L6 – Differential Geometry of Curves

- We will discuss local properties of curves independent of a possible embedding into a surface
- Topics to be covered including:
  - Parametric curves and arc length
  - Principal normal and curvature
  - Binormal vector and torsion
  - Frenet-Serret formulae

Parametric Curves

- A curve in R^3 is given by the parametric representation

$$\boldsymbol{r} = \boldsymbol{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad t \in [a, b] \subset R$$

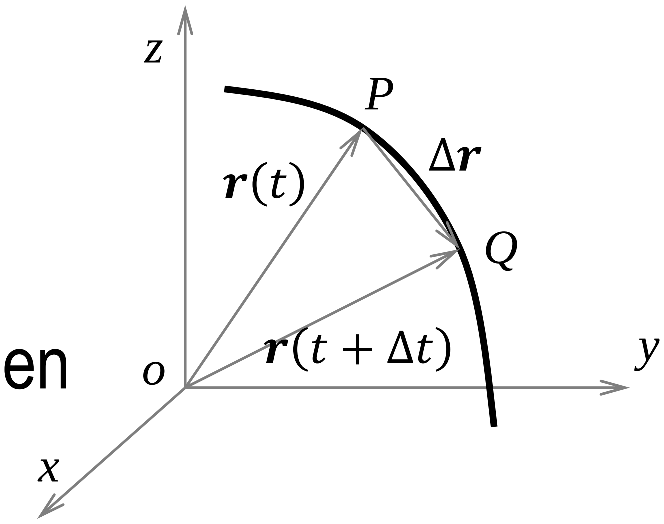
where $x(t)$, $y(t)$ and $z(t)$ are differentiable functions of t.

- A curve r(t) that satisfies

$$\dot{\boldsymbol{r}}(t) = \begin{bmatrix} \dot{x}\,(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} \neq 0 \qquad t \in [a, b]$$

is called a **regular** curve

Arc Length



- Considering a segment of $\boldsymbol{r}(t)$ between P and Q (i.e., as points $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t+\Delta t)$ respectively), its length Δs can be approximated as

$$\Delta s \approx \|\Delta \boldsymbol{r}\| = \|\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)\|$$

$$\approx \left\| \frac{d\boldsymbol{r}}{dt}\Delta t + \frac{d^2\boldsymbol{r}}{dt^2}(\Delta t)^2 \right\| \text{ (by Taylor expansion)}$$

$$\approx \left\| \frac{d\boldsymbol{r}}{dt} \right\| \Delta t$$

to the first order approximation.

- As point *Q* approaches *P* on the curve (i.e., $\Delta t \to 0$), the length Δs becomes the differential arc length of the curve as

$$ds = \left\| \frac{d\boldsymbol{r}}{dt} \right\| dt = \|\dot{\boldsymbol{r}}\| dt = \sqrt{\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}} dt$$

- The length of the curve can be computed as

$$s(t) = \int_{t_0}^{t} ds = \int_{t_0}^{t} \sqrt{\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}} dt$$

$$= \int_{t_0}^{t} \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} dt$$

- The vector $d\boldsymbol{r}/dt$ is called the tangent vector at point *P*, whose magnitude is derived from above as $\|\dot{\boldsymbol{r}}\| = \frac{ds}{dt}$

- Hence the **unit tangent vector** become

$$\boldsymbol{t} = \frac{\dot{\boldsymbol{r}}}{\|\dot{\boldsymbol{r}}\|} = \frac{d\boldsymbol{r}/_{dt}}{ds/_{dt}} = \frac{d\boldsymbol{r}}{ds} \equiv \boldsymbol{r}'$$

- We list some useful formulae of derivatives between s and t below.

$$\dot{s} = \frac{ds}{dt} = \|\dot{\boldsymbol{r}}\| = \sqrt{\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}} \qquad \ddot{s} = \frac{d\dot{s}}{dt} = \frac{\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}}}{\sqrt{\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}}}$$

$$t' = \frac{dt}{ds} = \frac{1}{\|\dot{\boldsymbol{r}}\|} = \frac{1}{\sqrt{\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}}} \qquad t'' = \frac{dt'}{ds} = -\frac{\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}}}{(\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}})^2}$$

Regularity of Parametric Curves

- A point $\boldsymbol{r}(t)$ is defined as the regular point if $\dot{\boldsymbol{r}}(t) \neq 0$; otherwise, it is called a singular point.
- A parameterization $\boldsymbol{r} = \boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big)^T$ of a curve defined in the interval I is called an allowable representation if it satisfies:
  - The mapping: $t \to \boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big)^T$ is one-to-one;
  - The vector function $\boldsymbol{r} = \boldsymbol{r}(t)$ is of class $r \geq 1$ in the interval I;
  - $||\dot{\boldsymbol{r}}(t)|| \neq 0$ for all $t \in I$.

Such a curve is called a regular curve.

Regularity of Implicit Curves

- A point (x_0,y_0) of a planar implicit curve $f(x,y)=0$ is said to be singular if $f(x_0,y_0) = f_x(x_0,y_0) = f_y(x_0,y_0) = 0$.
- Applying differentiation to an implicit curve, we can have

$$\mathrm{d}f = f_\mathrm{x}\,\mathrm{d}x + f_\mathrm{y}\,\mathrm{d}y = 0 \qquad \text{(as } f(x,y) = 0\text{)}$$

$$\Rightarrow \quad \mathrm{d}y / \mathrm{d}x = - f_\mathrm{x} / f_\mathrm{y} \quad \text{(if } f_y \neq 0\text{)}$$

- The tangent vector on the implicit curve is $\pm(f_y, -f_x)^T$; hence the unit tangent vector is:

$$\boldsymbol{t} = \pm\frac{(f_y, -f_x)^T}{\sqrt{f_x^2 + f_y^2}}$$

Implicit Space Curve

- An implicit space curve in 3D is defined as the intersection of two implicit surfaces

$$\begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}$$

- The normal vectors of these implicit surface are: $\nabla f, \nabla g$
- The unit tangent vector is

$$t = \pm \frac{\nabla f \times \nabla g}{\|\nabla f \times \nabla g\|}$$

with $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T$.

Principal Normal and Osculating Plane

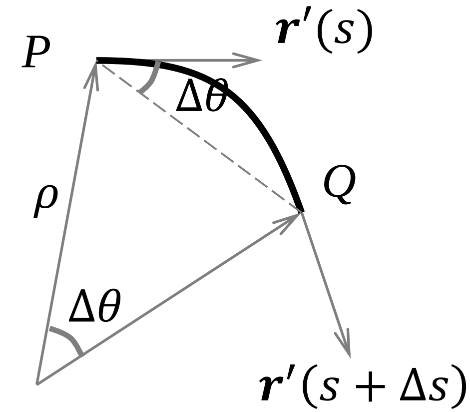
- If $\boldsymbol{r}(s)$ is an arc length parameterized curve, $\boldsymbol{r}'(s)$ is a unit vector with $\boldsymbol{r}' \cdot \boldsymbol{r}' = 1$
- Differentiating this, we obtain $\boldsymbol{r}' \cdot \boldsymbol{r}'' = 0$ (i.e., $\boldsymbol{r}' \perp \boldsymbol{r}''$)
- The unit vector

$$\boldsymbol{n} = \frac{\boldsymbol{r}''(s)}{||\boldsymbol{r}''(s)||} = \frac{\boldsymbol{t}'(s)}{||\boldsymbol{t}'(s)||}$$

is called the unit principal normal vector at s.

- The plane determined by $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$ is called the osculating plane at s.

Curvature



- From the right, we have

$$\|\boldsymbol{r}'(s+\Delta s)-\boldsymbol{r}'(s)\| = \Delta\theta \text{ when } \Delta s \to 0$$

$$\Rightarrow \|\boldsymbol{r}''(s)\| = \lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s} = \lim_{\Delta s\to 0}\frac{\Delta\theta}{\rho\Delta\theta} = \frac{1}{\rho} = \kappa$$

κ is called the curvature and its reciprocal ρ is called radius of curvature at s.

- It follows that: $\boldsymbol{r}'' = \boldsymbol{t}' = \kappa\boldsymbol{n}$
- The vector $\boldsymbol{\kappa} = \boldsymbol{r}'' = \boldsymbol{t}'$ is called the curvature vector.

Non-Arc-Length Parameterized Curve

$$\dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \boldsymbol{t}\nu$$

- $\nu = ds/dt$ defining the parametric speed, that is

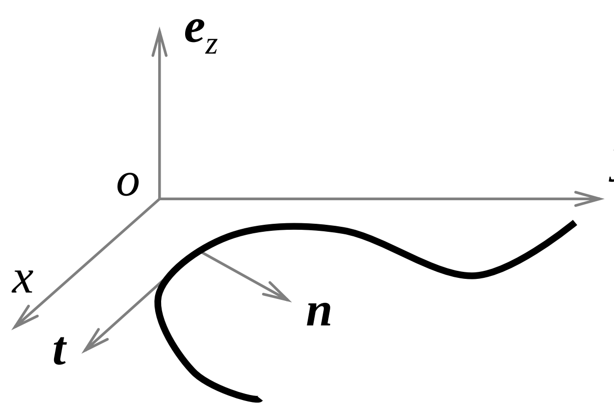
$$||\dot{\boldsymbol{r}}|| = \nu = ds/dt$$

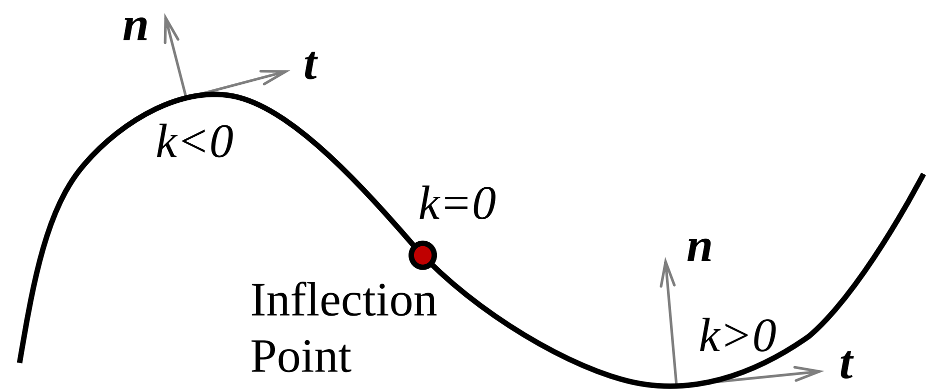
$$\ddot{\boldsymbol{r}} = \frac{d}{dt}(\boldsymbol{t}\nu) = \nu^2\frac{d\boldsymbol{t}}{ds} + \boldsymbol{t}\frac{d\nu}{dt} = \kappa\nu^2\boldsymbol{n} + \boldsymbol{t}\frac{d\nu}{dt}$$

- Then, we have

$$\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}} = \kappa\nu^3\boldsymbol{t} \times \boldsymbol{n}$$

As $\boldsymbol{t}\frac{d\nu}{dt}$ is parallel to $\dot{\boldsymbol{r}}$, the 2nd term eliminated.





- For the planar curve, we can give the curvature k a sign by defining the normal vector such that $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{e}_z)$ for a right-hand screw, where $\boldsymbol{e}_z = (0, 0, 1)^{\mathrm{T}}$.
- According to this, we have

$$\boldsymbol{n} = \boldsymbol{e}_z \times \boldsymbol{t} = \frac{(-\dot{y}, \dot{x})^T}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

- Hence from $\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}$ we have

$$\kappa = \frac{(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}) \cdot \boldsymbol{e}_z}{v^3} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

Curvatures for Parametric & Implicit Curves

- For a space curve, we can have

$$\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}} = \kappa v^3 \boldsymbol{t} \times \boldsymbol{n} \quad \Rightarrow \quad \kappa = \frac{\|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}\|}{\|v\|^3} = \frac{\|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}\|}{\|\dot{\boldsymbol{r}}\|^3}$$

- For a planar curve $f(x,y)=0$, we have

$$\boldsymbol{n} = \boldsymbol{e}_z \times \boldsymbol{t} = \frac{(f_x, f_y)^T}{\sqrt{f_x^2 + f_y^2}} = \frac{\nabla f}{\|\nabla f\|}$$

- We now start to derive the formula for curvature as follows.

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

with dx/ds and dy/ds to be further determined.

$$\boldsymbol{r}'' = \boldsymbol{t}' = \kappa\boldsymbol{n} = \kappa\frac{\nabla f}{\|\nabla f\|} \qquad \boldsymbol{t} = \frac{(f_y, -f_x)^T}{\sqrt{f_x^2 + f_y^2}} = \frac{(f_y, -f_x)^T}{\|\nabla f\|}$$

$$\boldsymbol{r}' = \boldsymbol{t} = \frac{d\boldsymbol{r}}{ds}$$

$\frac{dx}{ds} = \frac{f_y}{\|\nabla f\|}$ (the x−comp. of $\boldsymbol{t}$) $\frac{dy}{ds} = \frac{-f_x}{\|\nabla f\|}$ (the y−comp. of $\boldsymbol{t}$)

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \frac{1}{\|\nabla f\|}\left(f_y\frac{\partial f}{\partial x} - f_x\frac{\partial f}{\partial y}\right)$$

$$\frac{d}{ds} = \frac{1}{\|\nabla f\|}\left(f_y\frac{\partial}{\partial x} - f_x\frac{\partial}{\partial y}\right)$$

By $\frac{dt}{ds} = r'' = \kappa\frac{\nabla f}{\|\nabla f\|}$, we can obtain the curvature as

$$\kappa = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{(f_x^2 + f_y^2)^{3/2}}$$

Binormal Vector and Torsion

- Let's define a unit binormal vector $\boldsymbol{b}$ such that $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ form a right-handed screw:

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} \quad \boldsymbol{t} = \boldsymbol{n} \times \boldsymbol{b} \quad \boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{t}$$

- For the arbitrary speed curve, we can have

$$\boldsymbol{b} = \frac{\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}}{\|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}\|} \qquad \boldsymbol{b}' = \frac{d}{ds}(\boldsymbol{t} \times \boldsymbol{n}) = \frac{d\boldsymbol{t}}{ds} \times \boldsymbol{n} + \boldsymbol{t} \times \boldsymbol{n}'$$

$$= \kappa \boldsymbol{n} \times \boldsymbol{n} + \boldsymbol{t} \times \boldsymbol{n}' = \boldsymbol{t} \times \boldsymbol{n}'$$

- Since $\boldsymbol{n}$ is a unit vector $(\boldsymbol{n} \cdot \boldsymbol{n} = 1)$, we have $\boldsymbol{n} \cdot \boldsymbol{n}' = 0$ indicating that $\boldsymbol{n} \perp \boldsymbol{n}'$ (i.e., $\boldsymbol{n}'$ is // to the plane $(\boldsymbol{b}, \boldsymbol{t})$)

$$\Rightarrow \boldsymbol{n}' = \mu \boldsymbol{t} + \tau \boldsymbol{b}$$

Torsion

- By $\boldsymbol{b}' = \boldsymbol{t} \times \boldsymbol{n}'$, we could have
$$\boldsymbol{b}' = \boldsymbol{t} \times (\mu \boldsymbol{t} + \tau \boldsymbol{b}) = \tau(\boldsymbol{t} \times \boldsymbol{b}) = -\tau \boldsymbol{n}$$
- The coefficient τ is called the torsion and measures how much the curve deviates from the osculating plane $(\boldsymbol{n}, \boldsymbol{t})$
$$\tau = -\boldsymbol{n} \cdot \boldsymbol{b}'$$
$$= -\frac{\boldsymbol{r}''}{\kappa} \cdot \left(\boldsymbol{r}' \times \frac{\boldsymbol{r}''}{\kappa}\right)' = -\frac{\boldsymbol{r}''}{\kappa} \cdot \left(\boldsymbol{r}' \times \frac{\boldsymbol{r}'''}{\kappa}\right) = \frac{(\boldsymbol{r}'\boldsymbol{r}''\boldsymbol{r}''')}{\boldsymbol{r}'' \cdot \boldsymbol{r}''}$$
with $(\boldsymbol{r}'\boldsymbol{r}''\boldsymbol{r}''') = \boldsymbol{r}' \cdot (\boldsymbol{r}'' \times \boldsymbol{r}''')$ a triple scalar product.
- The torsion for an *arbitrary speed* curve is given by
$$\tau = \frac{(\dot{\boldsymbol{r}}\ddot{\boldsymbol{r}}\dddot{\boldsymbol{r}})}{(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}) \cdot (\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}})}$$

Frenet-Serret Formulae

- From the above analysis, have: $\boldsymbol{t}' = \kappa\boldsymbol{n}$ & $\boldsymbol{b}' = -\tau\boldsymbol{n}$

$$\boldsymbol{n}' = (\boldsymbol{b}\times\boldsymbol{t})' = \boldsymbol{b}'\times\boldsymbol{t} + \boldsymbol{b}\times\boldsymbol{t}'$$

$$= -\tau\boldsymbol{n}\times\boldsymbol{t} + \boldsymbol{b}\times\kappa\boldsymbol{n} = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}$$

We then obtain => $\begin{pmatrix}\boldsymbol{t}'\\ \boldsymbol{n}'\\ \boldsymbol{b}'\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{pmatrix}$

- The equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are called intrinsic equations of the curve

Frenet-Serret Formulae (cont.)

- For arbitrary speed curve, the Frenet-Serret formulae are given by

$$\begin{pmatrix} \dot{\boldsymbol{t}} \\ \dot{\boldsymbol{n}} \\ \dot{\boldsymbol{b}} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ -\upsilon\kappa & 0 & \upsilon\tau \\ 0 & -\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}$$

  with $\upsilon = \frac{ds}{dt} = ||\dot{\boldsymbol{r}}||$ the parametric speed.

- **Asssignment**: Given the intrinsic equations of circular helix

  $\kappa(s) = \frac{\alpha}{c^2}$ and $\tau(s) = \frac{\beta}{c^2}$ where $c = \sqrt{\alpha^2 + \beta^2}$

  Please derive the parametric equations of circular helix.