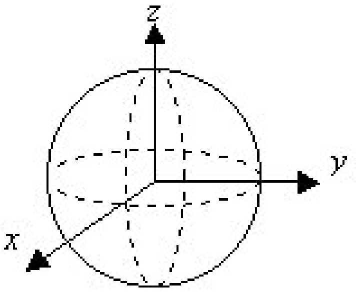
L7 – Polynomial / Spline Surfaces

- Ways to define and manipulate 3D surfaces, used by geometric modeling systems to store surfaces such as faces in a B-Rep
- Contents
  - Bi-linear Patch
  - Ruled Patch
  - Coons Patch
  - Bicubic Patch
  - Hermite Patch
  - Coons Patch with tangents
  - Bezier Patch
  - B-Spline Patch

Types of Surface Equations

- Implicit: describe a surface by equations relating to the *xyz*-coords
  - Advantages:
    - Compact; Easy to check if a point belongs to the surface
  - Disadvantages:
    - Difficult for surface evaluation
    - Difficult for partial surface definition (e.g., 1/4 of a sphere)
- Parametric: represent the *xyz*-coords as a function of two parameter
  - Advantages:
    - Easy for surface evaluation
    - Convenient for partial surface definition
    - Many others such as easy for manipulation

Implicit Surface Representations



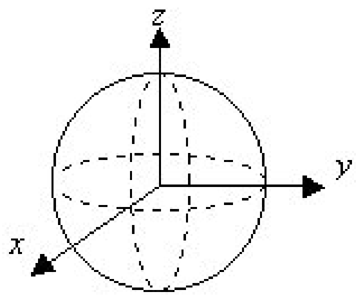
Implicit representation

$$x^2 + y^2 + z^2 - R^2 = 0$$

Problems:

- Surface evaluation. Say, to display this sohere on the screen, we need to approximate it by 1000 triangles, equally sized. Hard to do.
- Partial surface. To define an octant of the sphere, not the entire sphere which is closed. Difficult.

Parametric Surface Representations



Parametric representation

$x = R\cos\theta\sin\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\phi$ $\quad (0 \le \theta \le 2\pi)$ and $(0 \le \phi \le \pi)$.

- To compute 10000 equally spaced points on the sphere? Just evaluate the above equations at $(\theta_i, \phi_j) = ((i/1000)*2\pi, (j/1000)*2\pi)$, i = 0, 1, 2,…, 99, j = 1, 2, …, 99.
- To represent the sphere in the first octant? Just limit the range of (θ, ϕ) to $(0 \le \theta \le 0.5\pi, 0 \le \phi \le 0.5\pi)$.

Parametric Surfaces

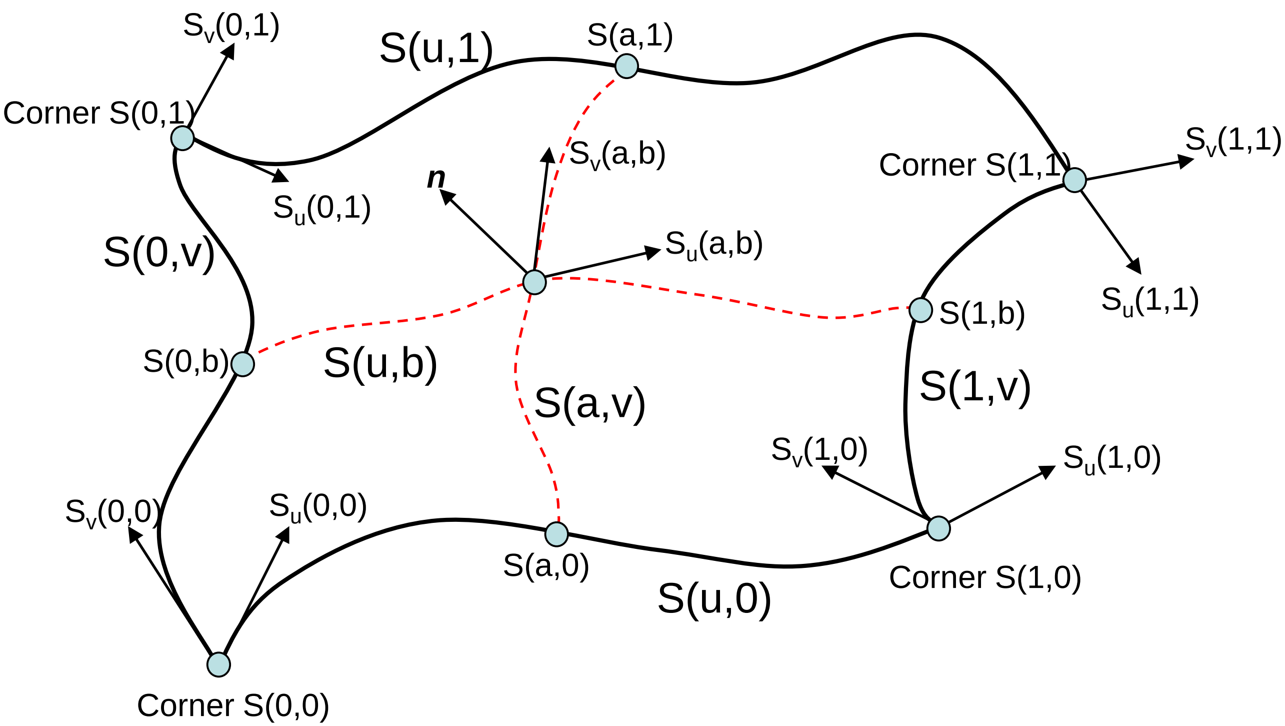
$S(u,v) = [x(u,v), y(u,v), z(u,v)]^T$ with $u_{min} < u < u_{max}$ and $v_{min} < v < v_{max}$

In most surfaces, the intervals for *u* and *v* are [0,1]. Surfaces can be modeled by a group of surface patches. A surface patch has the following boundary conditions:

- 4 corner vectors – **S**(0,0), **S**(0,1), **S**(1,0), **S**(1,1)
- 8 tangent vectors – 2 at each corner, $\mathbf{S}_u(u,v)$, $\mathbf{S}_v(u,v)$
- 4 twist vectors at the corners - $\mathbf{S}_{uv}(u,v)$
- 4 boundary curves – $u = 0$, $u = 1$, $v = 0$, $v = 1$.

Parametric Surfaces (cont.)

Basic Terminologies of Parametric Surface S(u,v)



$$S_u(u,v) = \left.\frac{\partial S}{\partial u}\right|_{(u,v)}$$

$$S_v(u,v) = \left.\frac{\partial S}{\partial u}\right|_{(u,v)}$$

$$\mathbf{n} = \frac{S_u(a,b) \times S_v(a,b)}{\left\| S_u(a,b) \times S_v(a,b) \right\|}$$

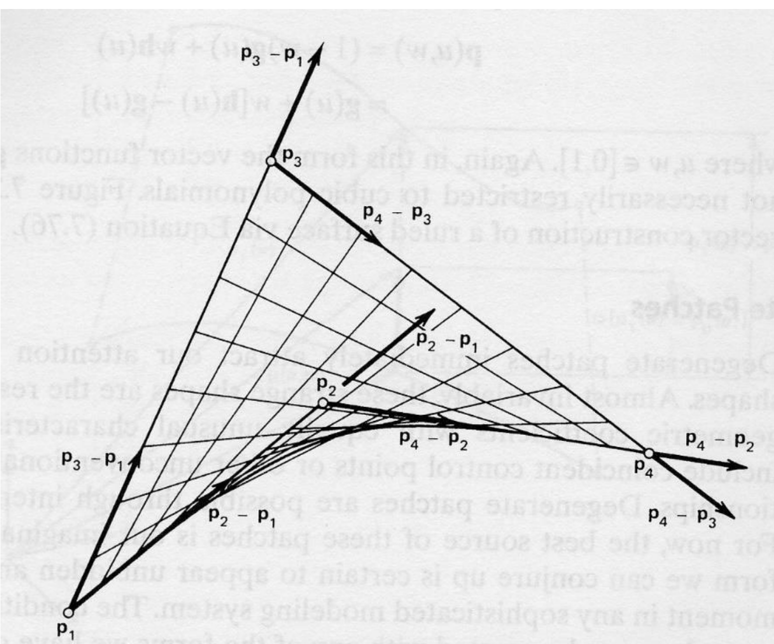
1. S(u,b) and S(a,v) are ***iso-parametric*** curves at v=b and u=a respectively.
2. **n** is the unit normal vector at (u,v) = (a,b).

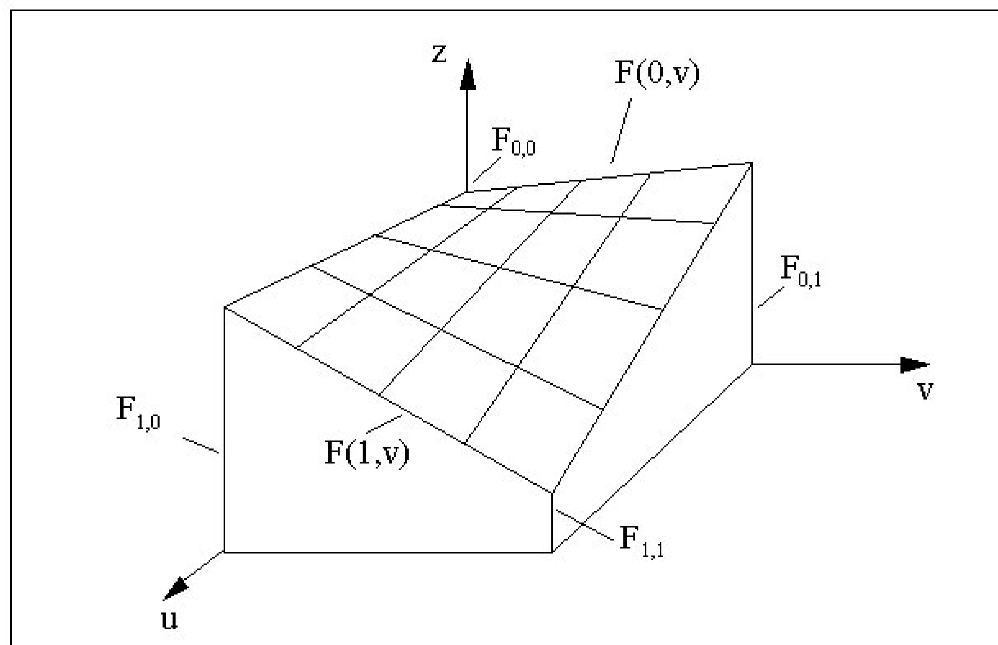
Classification of Surfaces

- Bi-linear Patch
- Ruled Patch
- Coons Patch
- Bicubic Patch
- Hermite Patch
- Coons Patch with tangents
- Bezier Patch
- B-Spline Patch
- Non-Uniform Rational B-Splines (NURBS)

Bi-linear Patch

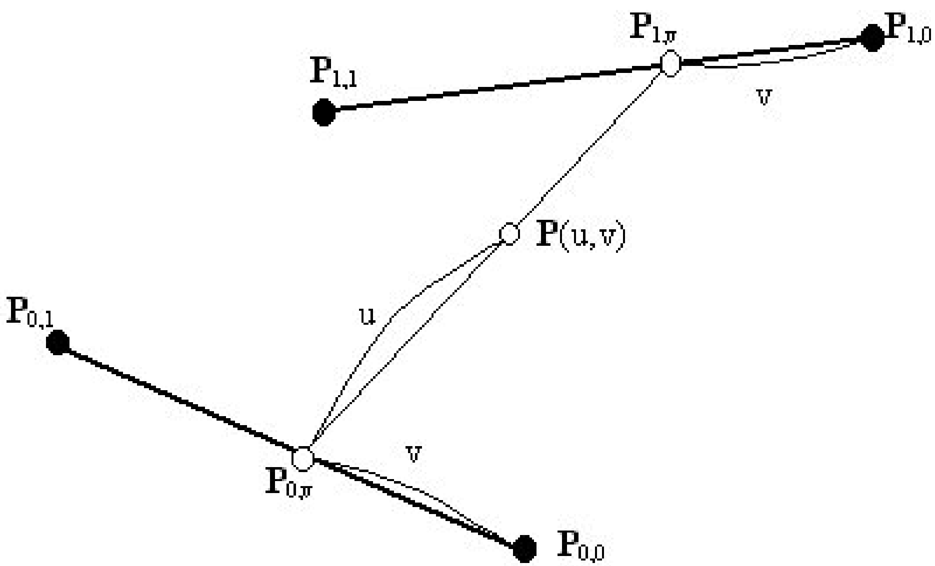
- The simplest surface defined by 4 points in space
  - Input: Four points $P_{0,0}$, $P_{0,1}$, $P_{1,0}$, $P_{1,1}$
  - Output: A surface S(*u*,*v*) with four corners S(0,0), S(0,1), S(1,0), and S(1,1) at the four given points
- Example of a Bi-linear Surface





Bi-linear surface
(The simplest surface defined by 4 points)

1. Define two linear boundary curves along u = 0 and u = 1



$$\mathbf{P}_{0,v} = (1\text{-}v)\ \mathbf{P}_{0,0} + v\mathbf{P}_{0,1}$$

$$\mathbf{P}_{1,v} = (1\text{-}v)\ \mathbf{P}_{1,0} + v\mathbf{P}_{1,1}$$

2. Define a linear curve along u direction between $\mathbf{P}_{0,v}$ and $\mathbf{P}_{1,v}$.

$$\mathbf{P}(u, v) = (1\text{-}u)\ \mathbf{P}_{0,v} + u\mathbf{P}_{1,v}$$

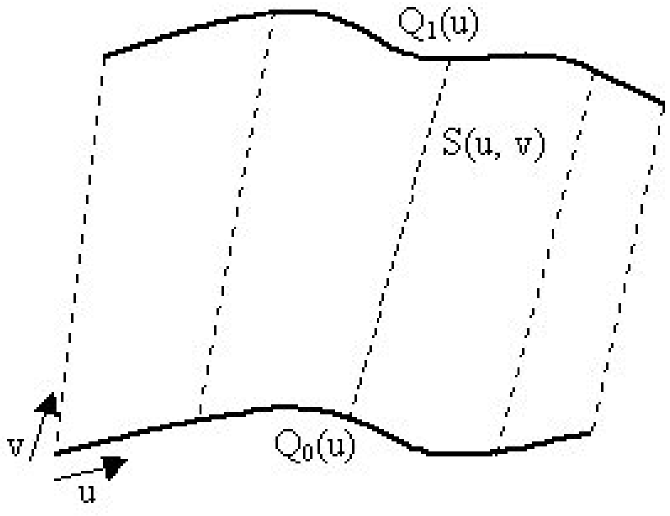
3. Merge 1 and 2 together.

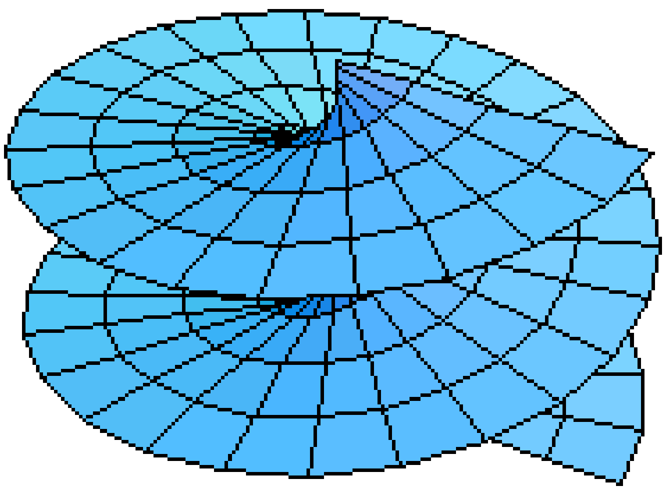
$$\mathbf{P}(u, v) = (1\text{-}u)(1\text{-}v)\mathbf{P}_{0,0} + u(1\text{-}v)\mathbf{P}_{1,0} + (1\text{-}u)v\mathbf{P}_{0,1} + uv\mathbf{P}_{1,1}$$

$$= \boldsymbol{BL}(u,v)\ (\mathbf{P}_{0,0}, \mathbf{P}_{1,0}, \mathbf{P}_{0,1}, \mathbf{P}_{1,1})$$

Ruled Surface

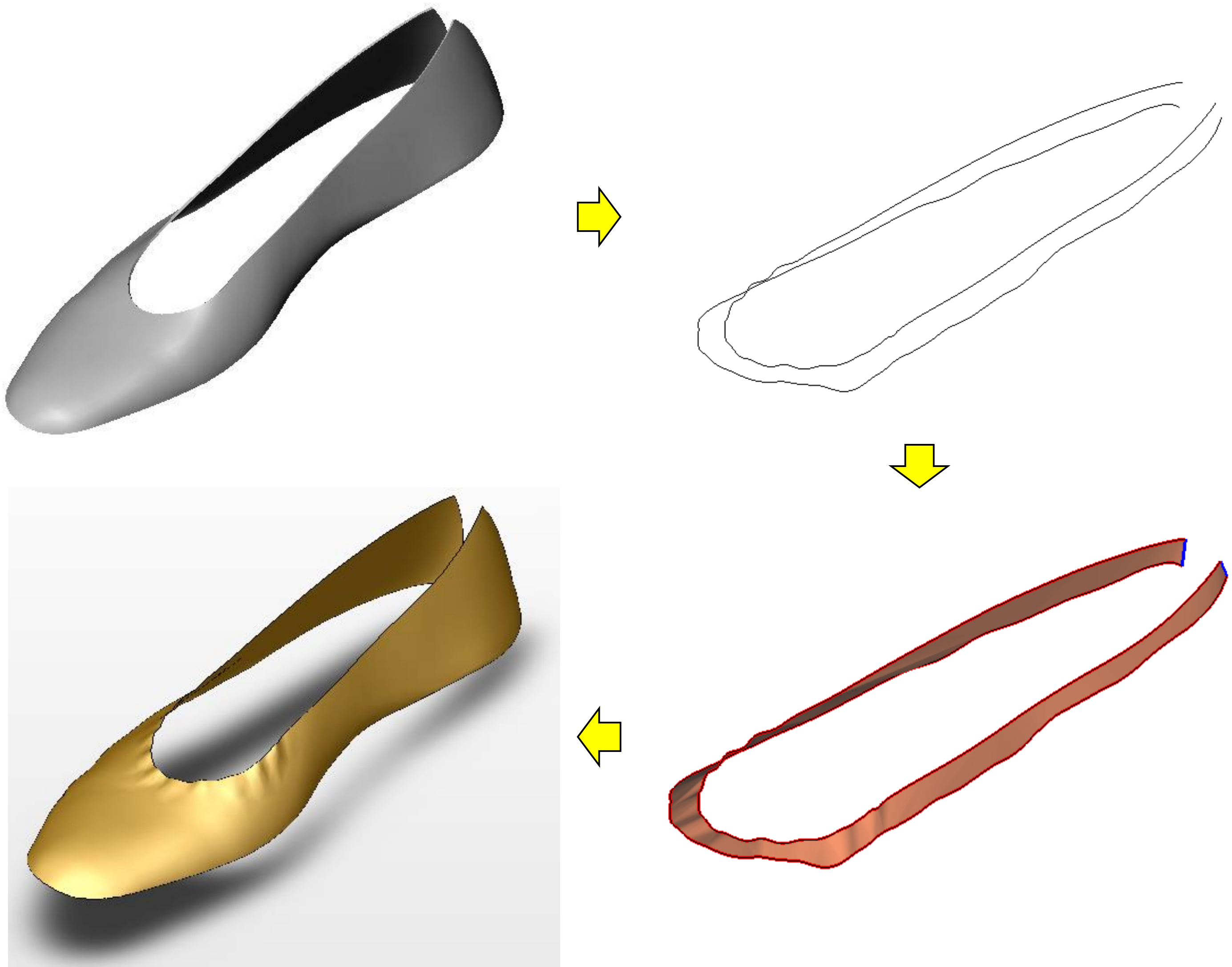
- The simplest surface defined by two curves
  - Input: Two curves $Q_0(u)$ and $Q_1(u)$ $(0 \leq u \leq 1)$
  - Output: A surface $S(u,v)$ with its two boundary curves $S(u,0)$ and $S(u,1)$ identical to $Q_0(u)$ and $Q_1(u)$ respectively
- Definition & Example





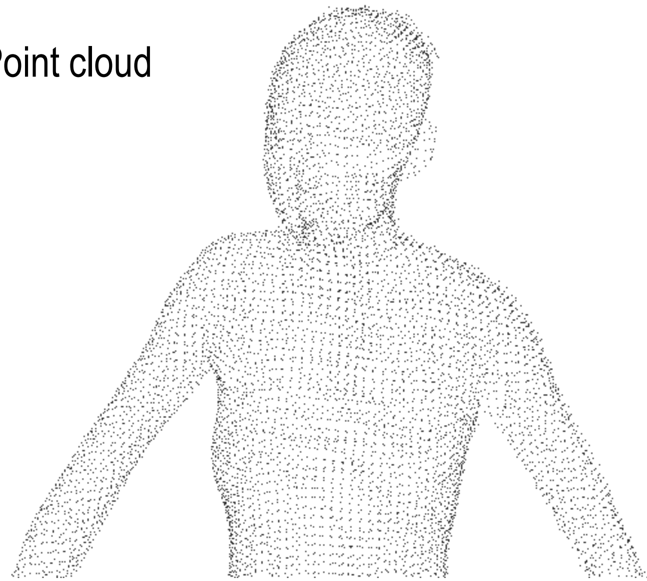
$S(u,v) = (1\text{-}v)\bullet Q_0(u) + v\bullet Q_1(u)$

Other Examples of Ruled Surface

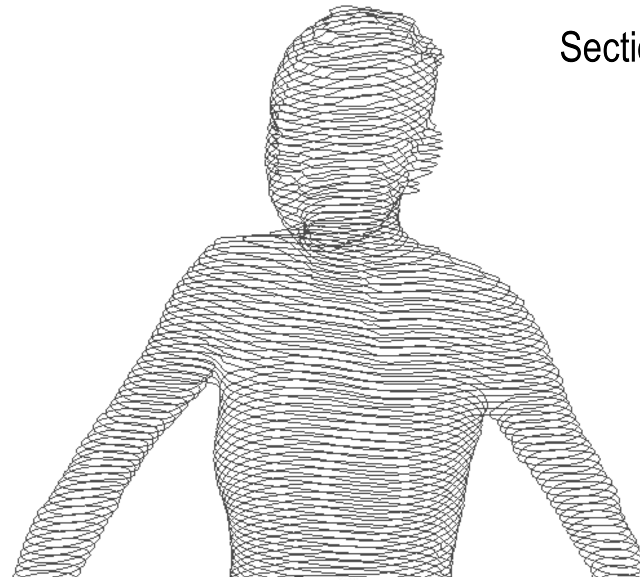


Other Examples of Ruled Surface

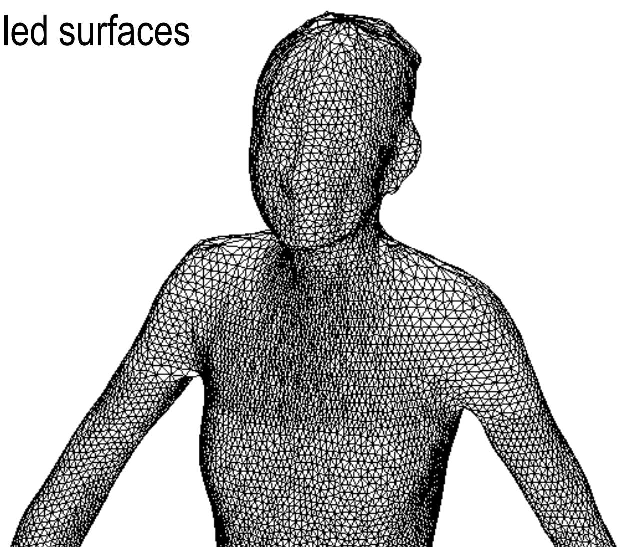
Point cloud



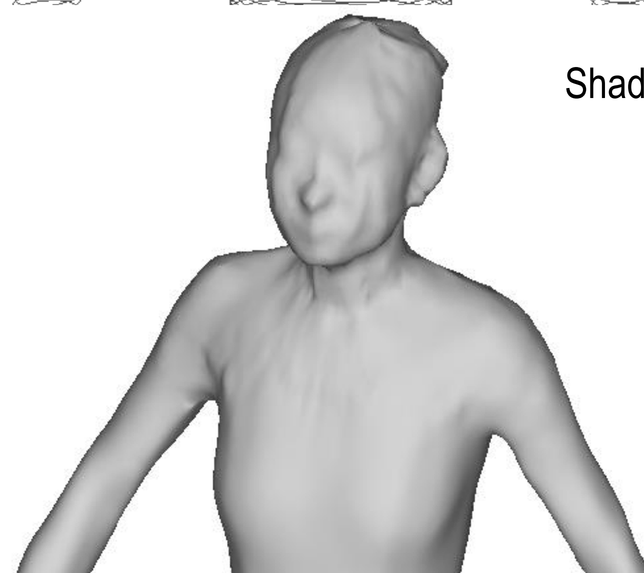
Section curves



Ruled surfaces



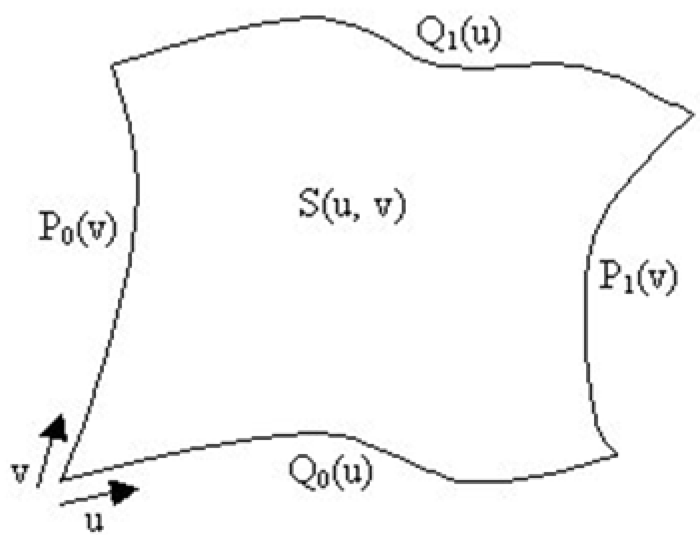
Shaded image



Coons Patch

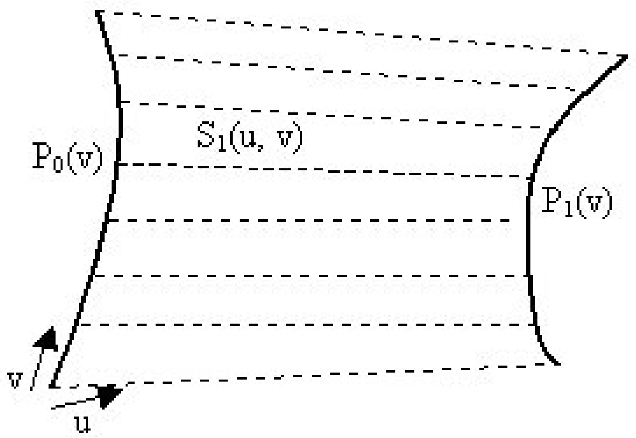
- Simplest surface defined by four curves $Q_0(u)$, $Q_1(u)$, $P_0(v)$, $P_1(v)$
- Input and output

Given four boundary curves $\mathbf{Q}_0(u)$, $\mathbf{Q}_1(u)$, $\mathbf{P}_0(v)$ and $\mathbf{P}_1(v)$ on which a patch $S(u, v)$ needs to be defined, such that the four boundary curves $S(u,0)$, $S(u,1)$, $S(0,v)$, and $S(1,v)$ are identical to the four curves $\mathbf{Q}_0(u)$, $\mathbf{Q}_1(u)$, $\mathbf{P}_0(v)$ and $\mathbf{P}_1(v)$ respectively. Bi-linear surface can be viewed as a special case of this in which the four input curves are all linear.



Defining a Coons Patch: Step 1 and Step 2

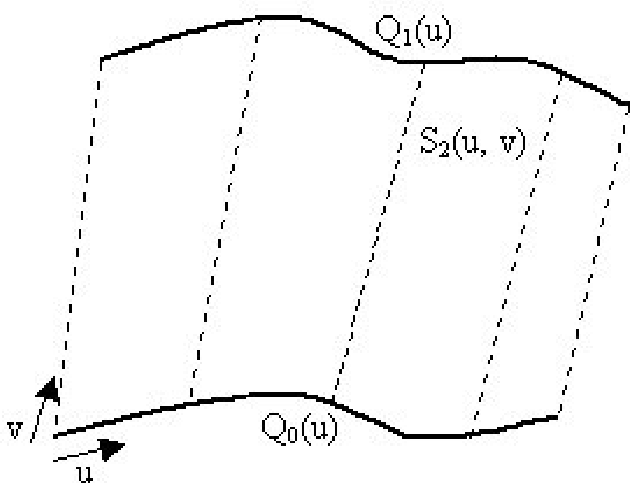
1. Define a ruled surface along u direction between $\mathbf{P}_0(v)$ and $\mathbf{P}_1(v)$.



$$\mathbf{S}_1(u,v) = (1-u)\mathbf{P}_0(v) + u\mathbf{P}_1(v)$$

This surface is bounded by $\mathbf{P}_0(v)$ at $u = 0$ and $\mathbf{P}_1(v)$ at $u = 1$, as desired. However, the other pairs of boundary curves will be straight line segments, not the desired $\mathbf{Q}_0(u)$ and $\mathbf{Q}_1(u)$.

2. Define a ruled surface along v direction between $\mathbf{Q}_0(u)$ and $\mathbf{Q}_1(u)$.



$$\mathbf{S}_2(u,v) = (1-v)\mathbf{Q}_0(u) + v\mathbf{Q}_1(u)$$

This surface is bounded by $\mathbf{Q}_0(u)$ at $v = 0$ and $\mathbf{Q}_1(u)$ at $v = 1$, as desired. However, the other pairs of boundary curves will be straight line segments, not the desired $\mathbf{P}_0(v)$ and $\mathbf{P}_1(v)$.

Defining a Coons Patch: Step 3

3. Add two together and find the compensating patch.

$$\mathbf{P}(u,v) = \mathbf{S}_1(u,v) + \mathbf{S}_2(u,v)$$

At the boundary:

$$\mathbf{P}(0,v) = \mathbf{P}_0(v) + \underline{(1\text{-}v)\mathbf{Q}_0(0) + v\mathbf{Q}_1(0)}$$

$$\mathbf{P}(1,v) = \mathbf{P}_1(v) + \underline{(1\text{-}v)\mathbf{Q}_0(1) + v\mathbf{Q}_1(1)}$$

$$\mathbf{P}(u,0) = \mathbf{Q}_0(u) + \underline{(1\text{-}u)\mathbf{P}_0(0) + u\mathbf{P}_1(0)}$$

$$\mathbf{P}(u,1) = \mathbf{Q}_1(u) + \underline{(1\text{-}u)\mathbf{P}_0(1) + u\mathbf{P}_1(1)}$$

The underlined are unwanted. They are the line segments connecting the four corners. So if we define a bilinear patch on these four corners,

$$\mathbf{S}_3(u,v) = \boldsymbol{BL}(u,v)\ (\mathbf{Q}_0(0),\ \mathbf{Q}_0(1),\ \mathbf{Q}_1(0),\ \mathbf{Q}_1(1)),$$

and subtract it from $\mathbf{P}(u,v)$, the result is what we want, a ***Coons' patch***:

$$\boxed{\mathbf{S}(u,v) \ = \mathbf{P}(u,v) - \mathbf{S}_3(u,v) = \mathbf{S}_1(u,v) + \mathbf{S}_2(u,v) - \mathbf{S}_3(u,v).}$$

Generalized Coons Patch

The terms (1-u) and u (similarly for v and (1-v)) in the Coons patch are linear blending functions, as a direct result of the bi-linear construction in Step 1 and Step 2. If we replace them by a pair $\{\alpha_0(u), 1-\alpha_0(u)\}$ (similarly for v), where $a_0(u)$ can be any continuous function, the resulting surface still meets the boundary condition: $S(0,v) = P_0(v)$, $S(1,v) = P_1(v)$, $S(u,0) = Q_0(u)$, and $S(u,1) = Q_1(u)$.

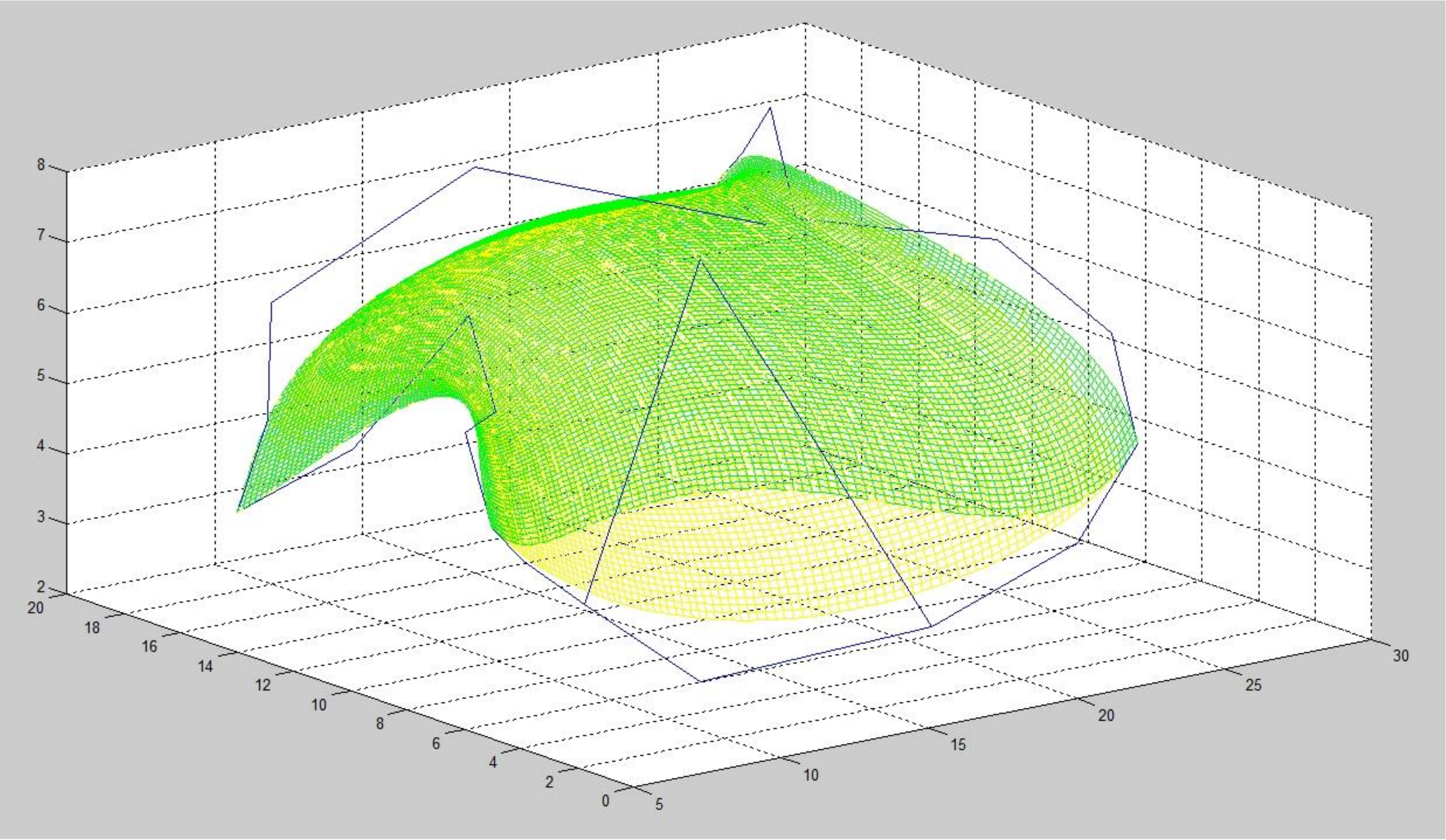
$$S(u,v) = \begin{bmatrix}(1-u) & u\end{bmatrix}\begin{bmatrix}P_0(v)\\ P_1(v)\end{bmatrix} + \begin{bmatrix}Q_0(u) & Q_1(u)\end{bmatrix}\begin{bmatrix}1-v\\ v\end{bmatrix} - \begin{bmatrix}(1-u) & u\end{bmatrix}\begin{bmatrix}Q_0(0) & Q_1(0)\\ Q_0(1) & Q_1(1)\end{bmatrix}\begin{bmatrix}1-v\\ v\end{bmatrix}$$

$$(1-u) \rightarrow \alpha_0(u)$$
$$u \rightarrow 1-\alpha_0(u)$$

$$S(u,v) = \begin{bmatrix}\alpha_0(u) & 1-\alpha_0(u)\end{bmatrix}\begin{bmatrix}P_0(v)\\ P_1(v)\end{bmatrix} + \begin{bmatrix}Q_0(u) & Q_1(u)\end{bmatrix}\begin{bmatrix}\alpha_0(v)\\ 1-\alpha_0(v)\end{bmatrix} - \begin{bmatrix}\alpha_0(u) & 1-\alpha_0(u)\end{bmatrix}\begin{bmatrix}Q_0(0) & Q_1(0)\\ Q_0(1) & Q_1(1)\end{bmatrix}\begin{bmatrix}\alpha_0(v)\\ 1-\alpha_0(v)\end{bmatrix}$$

Example: $\alpha_0(u) = 1 - 3u^2 + 2u^3$

Coons Patch Example



Modifying one of the four boundary curves will cause the corresponding Coons patch to alter only the interior of the surface and the curve itself – the other three boundary curves remain unchanged.

Bi-cubic Patch

A bicubic patch is a surface represented by an equation in polynomial form of degree 3 in the parameters u and v as in:

$$\mathbf{S}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij} u^i v^j \qquad (0 \le u \le 1, 0 \le v \le 1)$$

Or , in matrix form:

$$\mathbf{S}(u,v) = [1\ u\ u^2\ u^3] \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix} \qquad (0 \le u \le 1, 0 \le v \le 1)$$

where each $\boldsymbol{a}_{ij}$ is an algebraic vector with x, y, and z components.

Example:

$$\mathbf{S}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} = \begin{bmatrix} 0.5 + 2v + v^3 - uv^2 - 4u^3v^3 \\ 3v^2 + 5u^2v \\ 1 + 2.5u - 1.5u^2v^3 \end{bmatrix}$$

$$\boldsymbol{a}_{00} = \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{a}_{01} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{a}_{02} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \boldsymbol{a}_{03} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{a}_{10} = \begin{bmatrix} 0 \\ 0 \\ 2.5 \end{bmatrix}, \boldsymbol{a}_{12} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{a}_{21} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \boldsymbol{a}_{23} = \begin{bmatrix} 0 \\ 0 \\ -1.5 \end{bmatrix}, \boldsymbol{a}_{33} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$$

All other $\boldsymbol{a}_{ij}$ are $[0\ 0\ 0]^T$.

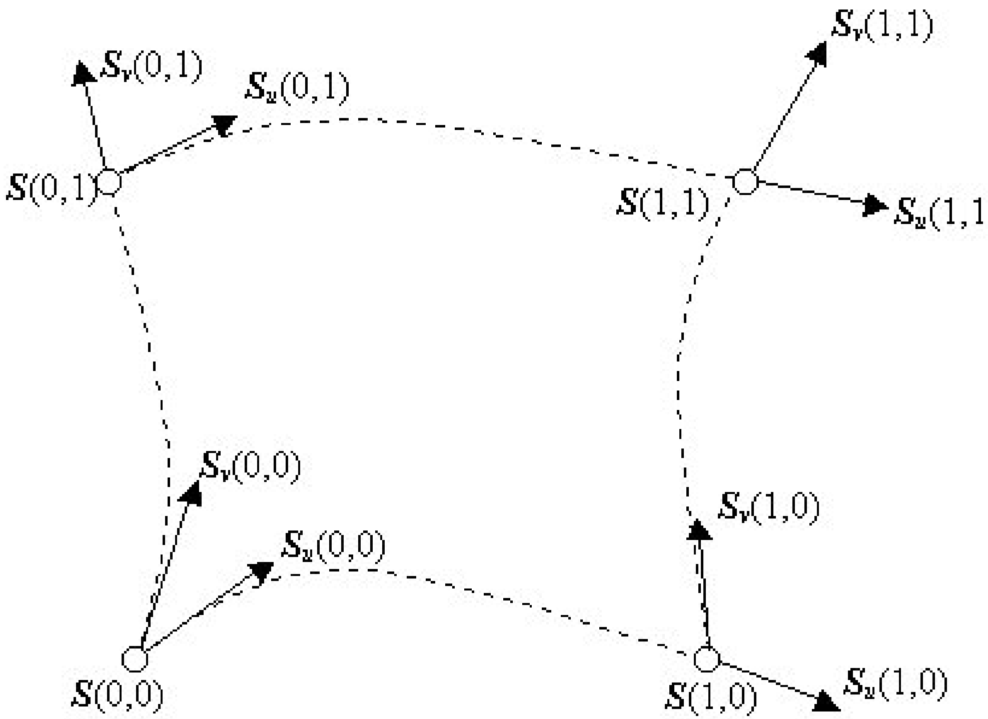
Hermite Patch

- Similar to Hermite curve, non-intuitive algebraic coefficients $\boldsymbol{a}_{ij}$ need to be replaced by geometric coefficients like corner points and tangents. There are 16 unknowns $\boldsymbol{a}_{ij}$, so we need 16 boundary vectors in order to find them.
- 12 intuitive vectors
- 4 more boundary vectors (twist vectors)

  (usually set to zero vectors if difficult to decide)
- Compute the Hermite form

1. Compute the derivatives of the bicubic surface
2. Plug in the 16 boundary vectors and solve the linear equations for aij
3. Rearrange into the Hermite Form

First 12 Boundary Conditions in a Patch



- The four corner points $\mathbf{S}(0, 0)$, $\mathbf{S}(0, 1)$, $\mathbf{S}(1, 0)$, and $\mathbf{S}(1, 1)$.
- The four tangent vectors along u direction at the four corners: $\mathbf{S}_u(0, 0)$, $\mathbf{S}_u(0, 1)$, $\mathbf{S}_u(1, 0)$, and $\mathbf{S}_u(1, 1)$.
- The four tangent vectors along v direction at the four corners: $\mathbf{S}_v(0, 0)$, $\mathbf{S}_v(0, 1)$, $\mathbf{S}_v(1, 0)$, and $\mathbf{S}_v(1, 1)$.

Four more boundary conditions in a patch

The 2^{nd}-order cross-derivative is defined as:

$$\mathbf{S}_{uv}(u, v) = \frac{\partial^2 S(u,v)}{\partial u \partial v}.$$

- The 2^{nd}-order cross-derivatives at the four corners: $\mathbf{S}_{uv}(0, 0)$, $\mathbf{S}_{uv}(0, 1)$, $\mathbf{S}_{uv}(1, 0)$, and $\mathbf{S}_{uv}(1, 1)$.

- Compute the Hermite form

1. Compute the derivatives of the bicubic surface
2. Plug in the 16 boundary vectors and solve the linear equations for $\boldsymbol{a}_{ij}$
3. Rearrange into the Hermite Form

Derivatives of Bi-cubic Patch

$$\mathbf{S}(u,v) = [1\ u\ u^2\ u^3]\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

$$\mathbf{S}_u(u,v) = \left([1\ u\ u^2\ u^3]\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}\right)_u = [0\ 1\ 2u\ 3u^2]\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

$$\mathbf{S}_v(u,v) = \left([1\ u\ u^2\ u^3]\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}\right)_v = [1\ u\ u^2\ u^3]\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2v \\ 3v^2 \end{bmatrix}$$

$$\mathbf{S}_{uv}(u,v) = \left([0\ 1\ 2u\ 3u^2]\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}\right)_v = [0\ 1\ 2u\ 3u^2]\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2v \\ 3v^2 \end{bmatrix}$$

Definition of Hermite Patch

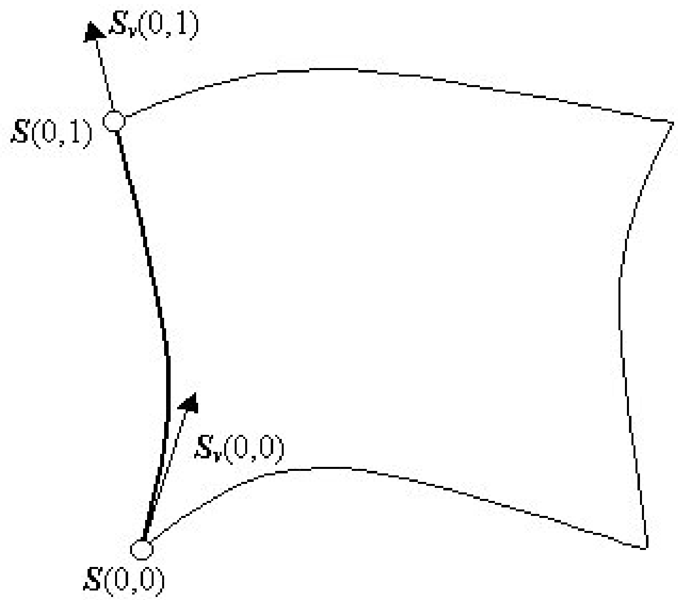
$$\mathbf{S}(u,v) = [f_0(u)\ \ f_1(u)\ \ f_2(u)\ \ f_3(u)] \begin{bmatrix} S(0,0) & S(0,1) & S_v(0,0) & S_v(0,1) \\ S(1,0) & S(1,1) & S_v(1,0) & S_v(1,1) \\ S_u(0,0) & S_u(0,1) & S_{uv}(0,0) & S_{uv}(0,1) \\ S_u(1,0) & S_u(1,1) & S_{uv}(1,0) & S_{uv}(1,1) \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \\ f_2(v) \\ f_3(v) \end{bmatrix}$$

$$(0 \le u \le 1, 0 \le v \le 1)$$

where the blending functions $f_0(u)$, $f_1(u)$, $f_2(u)$, and $f_3(u)$ are Hermite functions:

$$f_0(u) = 1 - 3u^2 + 2u^3$$
$$f_1(u) = 3u^2 - 2u^3$$
$$f_2(u) = u - 2u^2 + u^3$$
$$f_3(u) = -u^2 + u^3.$$

Boundary curves are Hermite



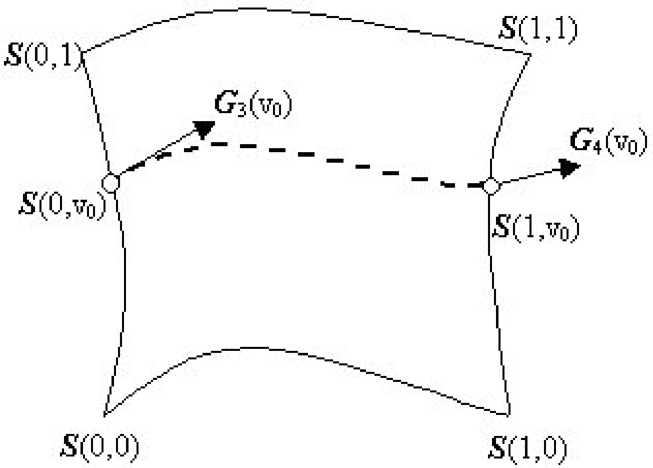
Each of the four boundary curves is a Hermite curve.

Example: **S**(0,v). After substituting 0 for u into S(u,v):

$$\mathbf{S}(0,v) = [f_0(v) \;\; f_1(v) \;\; f_2(v) \;\; f_3(v)] \begin{bmatrix} S(0,0) \\ S(0,1) \\ S_v(0,0) \\ S_v(0,1) \end{bmatrix}.$$

S(0, 0) and **S**(0, 1) are the two corners and $\mathbf{S}_v(0, 0)$ and $\mathbf{S}_v(0, 1)$ are the tangent vectors of the patch along the v direction. Notice that $\frac{\partial S(0,v)}{\partial v}\Big|_{v=0} = \mathbf{S}_v(0, 0)$, and $\frac{\partial S(0,v)}{\partial v}\Big|_{v=1} = \mathbf{S}_v(0, 1)$.

Iso-parametric Curves are Hermite Curves



Any iso-parametric curve $\mathbf{S}(u, v_0)$ or $\mathbf{S}(u_0, v)$is a Hermite.

Example: $\mathbf{S}(u, v_0)$. Substituting $v = v_0$ leads to:

$$\mathbf{S}(u, v_0) = [f_0(u) \;\; f_1(u) \;\; f_2(u) \;\; f_3(u)] \begin{bmatrix} S(0,v_0) \\ S(1,v_0) \\ G_3(v_0) \\ G_4(v_0) \end{bmatrix}.$$

where:

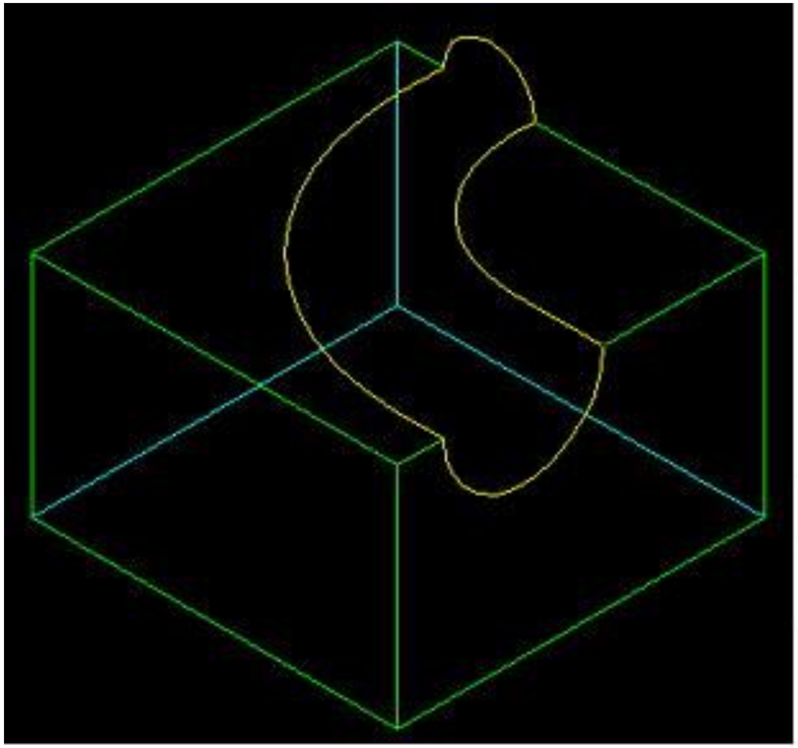
$$G_3(v_0) = \mathbf{S}_u(0,0)f_0(v_0) + \mathbf{S}_u(0,1)f_1(v_0) + \mathbf{S}_{uv}(0,0)f_2(v_0) + \mathbf{S}_{uv}(0,1)f_3(v_0)$$

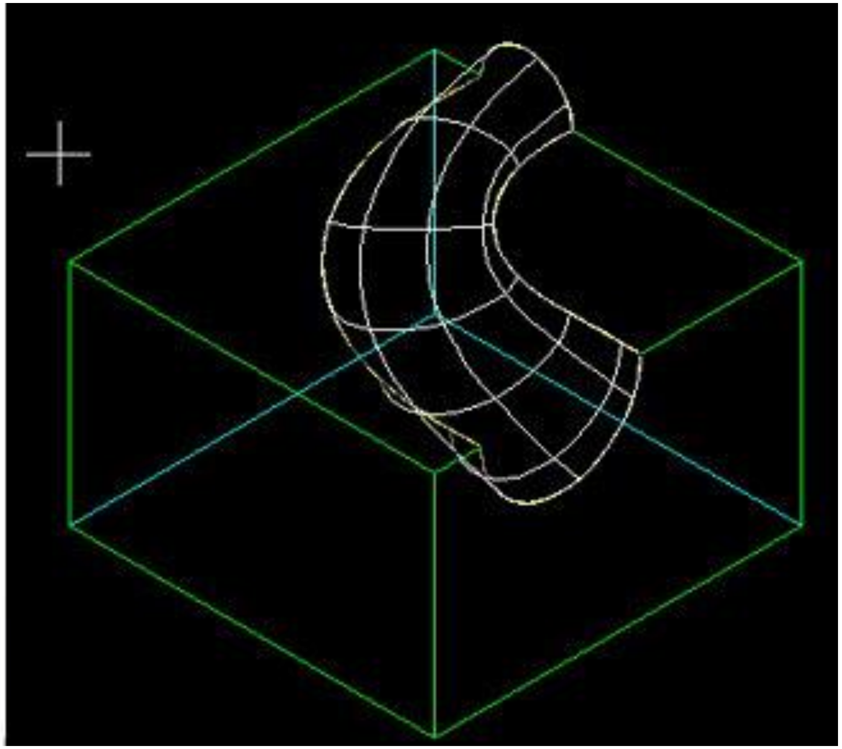
$$G_4(v_0) = \mathbf{S}_u(1,0)f_0(v_0) + \mathbf{S}_u(1,1)f_1(v_0) + \mathbf{S}_{uv}(1,0)f_2(v_0) + \mathbf{S}_{uv}(1,1)f_3(v_0).$$

Note that:

$$\left.\frac{\partial S(u,v)}{\partial u}\right|_{(0,v_0)} = G_3(v_0) \qquad \left.\frac{\partial S(u,v)}{\partial u}\right|_{(1,v_0)} = G_4(v_0).$$

Example of Hermite Patch





Drawbacks of Hermite Patch: It is not easy and not intuitive to predict surface shape according to changes in magnitude of the tangents (partial derivatives) at the four corners. In addition, the four cross-derivatives $S_{uv}(0, 0)$, $S_{uv}(0, 1)$, $S_{uv}(1, 1)$ and $S_{uv}(1, 0)$ most of time are not known.

Bezier Surface

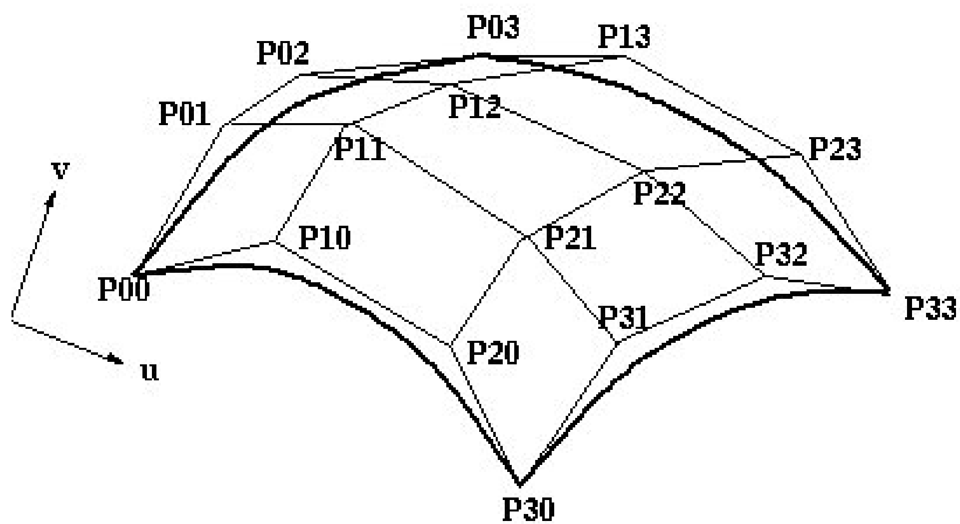
- Definition:

$$\mathbf{S}(\mathrm{u}, \mathrm{v}) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,n}(u) B_{j,m}(v) \mathbf{P}_{\mathrm{ij}} \qquad (0 \le u \le 1 \,, 0 \le v \le 1)$$

$\boldsymbol{p}_{ij}$: the control points that form a (n+1) by (m+1) control mesh

n: the degree of the surface in u direction

m: the degree of the surface in v direction



Properties of Bezier Surface

- The four control points $\mathbf{P}_{00}$, $\mathbf{P}_{0m}$, $\mathbf{P}_{n0}$, and $\mathbf{P}_{nm}$ lie on the surface and are its four corners $\mathbf{S}(0,0)$, $\mathbf{S}(0,1)$, $\mathbf{S}(1,0)$, and $\mathbf{S}(1,1)$ respectively.

- Any iso-parametric curve is a Bezier curve.

  ($\mathbf{S}(u_0, v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u_0)B_{j,m}(v)\,\mathbf{P}_{ij}$ is a Bezier curve, so is $\mathbf{S}(u, v_0)$, for any constant u_0 or v_0.)

- Convex Hull Property – The Bezier patch is inside the convex hull of its control points.

  (To prove: show that $\sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u)B_{j,m}(v) = 1$ for any u and v.)

- The partial derivative $\left.\frac{\partial S(u,v)}{\partial u}\right|_{(0,0)}$ is parallel to $(\mathbf{P}_{10} - \mathbf{P}_{00})$, and the partial derivative $\left.\frac{\partial S(u,v)}{\partial v}\right|_{(0,0)}$ is parallel to $(\mathbf{P}_{01} - \mathbf{P}_{00})$.

  (Similarly for the other three corners)

- The partial derivatives $\frac{\partial S(u,v)}{\partial u}$ and $\frac{\partial S(u,v)}{\partial v}$ are also Bezier surfaces themselves.

Evaluation of Bezier Surface

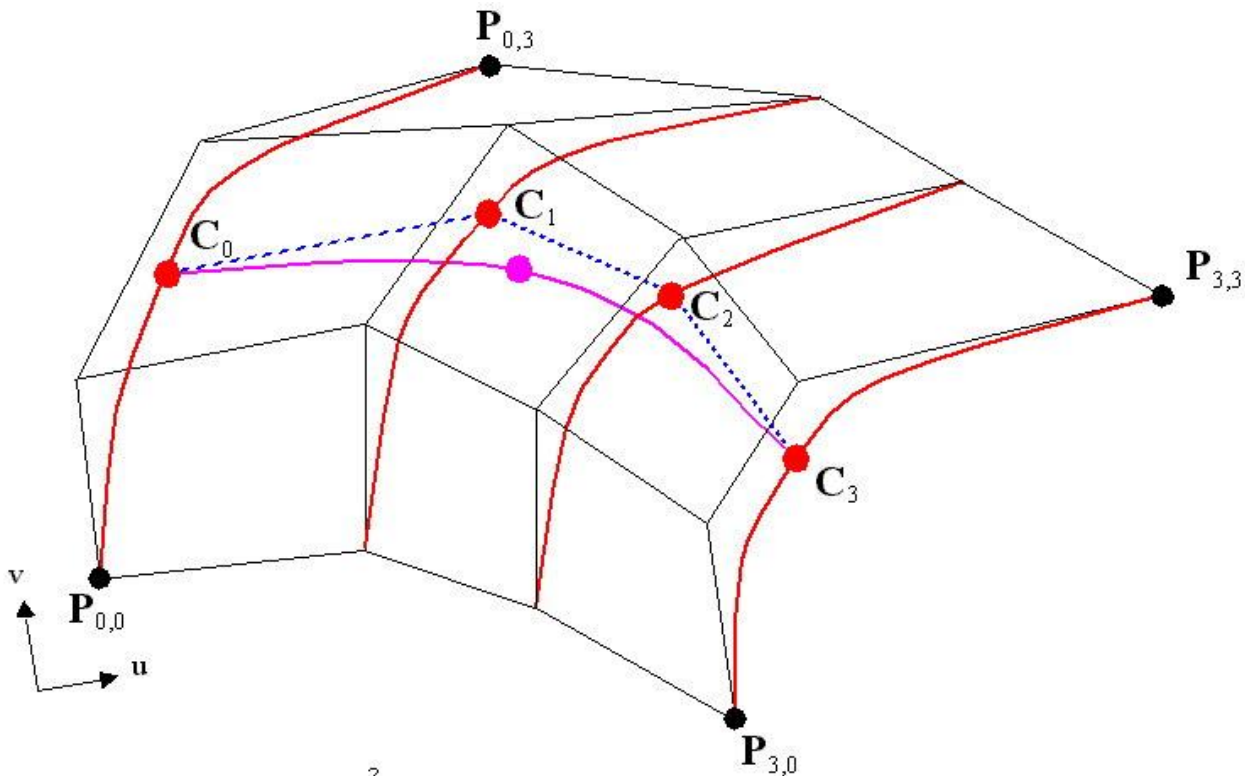
How to evaluate a Bezier patch at a point (u_0, v_0)? By applying the de Casteljau algorithm recursively.

$$\mathbf{S}(u_0, v_0) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u_0) B_{j,m}(v_0)\, \mathbf{P}_{ij}$$

$$= \left[\sum_{j=0}^{m} P_{0,j} B_{j,m}(v_0)\right] B_{0,n}(u_0) + \left[\sum_{j=0}^{m} P_{1,j} B_{j,m}(v_0)\right] B_{1,n}(u_0) + \ldots + \left[\sum_{j=0}^{m} P_{n,j} B_{j,m}(v_0)\right] B_{n,n}(u_0)$$

- Using de Casteljau algorithm to evaluate $C_i = \sum_{j=0}^{m} P_{i,j} B_{j,m}(v_0)$ (i=0,1,…,n)
- Using de Casteljau algorithm again to evaluate $\mathbf{S}(u_0, v_0) = \sum_{i=0}^{n} C_i B_{i,n}(u_0)$

Pictorial Illustration of Evaluating a Cubic Bezier Surface



$$\mathbf{C_i} = \sum_{j=0}^{3} B_{j,3} P_{i,j}(v_0)$$

$$\mathbf{S}(u_0, v_0) = \sum_{i=0}^{3} B_{i,3} C_i(u_0)$$

Drawbacks of Bezier Surface

- High degree
  - The degree is determined by the number of control points which tend to be large for complicated surfaces. This causes oscillation as well as increases the computation burden.
- Non-local modification control
  - When modifying a control point, the designer wants to see the shape change locally around the moved control point. In Bezier patch case, moving a control point affects the shape of the entire surface, and thus the portions on the surface not intended to change.
- Intractable linear equations
  - If we are interested in interpolation rather than just approximating a shape, we will have to compute control points from points on the surface. This leads to systems of linear equations, and solving such systems can be impractical when the degree of the surface is large.

B-Spline Surface

- Definition:

$$\mathbf{S}(\mathrm{u}, \mathrm{v}) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,k}(u) N_{j,l}(v)\, \mathbf{P}_{\mathrm{ij}} \qquad (s_{k-1} \le u \le s_{n+1}, \;\; t_{l-1} \le v \le t_{m+1})$$

$\boldsymbol{p}_{ij}$: the control points that form a (n+1) by (m+1) control mesh

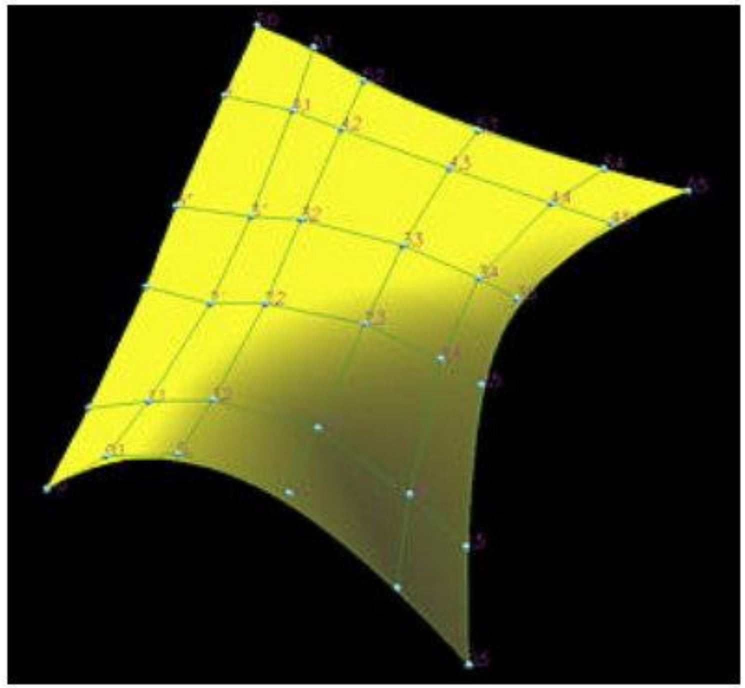
k: the order of the basis functions in u direction

l: the order of the basis functions in v direction

$U = \{s_0, \ldots, s_{n+k}\}$ is the knots vector in u-direction

$V = \{t_0, \ldots, t_{n+l}\}$ is the knots vector in v-direction

Example B-Spline Surface



n = m = 5

k = 3, knot vector (u-direction) U = {0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1}

l = 4, knot vector (v-direction) V = {0, 0, 0, 0, 0.33, 0.66, 1, 1, 1, 1}

Both U and V are non-periodic and uniform (the patch passes all the four corners).

Properties of B-Spline Surface

- The four control points $\mathbf{P}_{00}$, $\mathbf{P}_{0m}$, $\mathbf{P}_{n0}$, and $\mathbf{P}_{nm}$ lie on the surface and are its four corners $\mathbf{P}(s_{k-1}, t_{l-1})$, $\mathbf{P}(s_{k-1}, t_{m+1})$, $\mathbf{P}(s_{n+1}, t_{l-1})$, and $\mathbf{P}(s_{n+1}, t_{m+1})$ respectively. (For non-periodic only.)

- Convex Hull Property – The B-spline patch is inside the convex hull of its control points. ($\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,k}(u) N_{j,l}(v) = 1$ for all u and v.)

- Any isoparametric curve is a B-spline curve.
(Similar to that any isoparametric curve of a Bezier patch is a bezier curve)

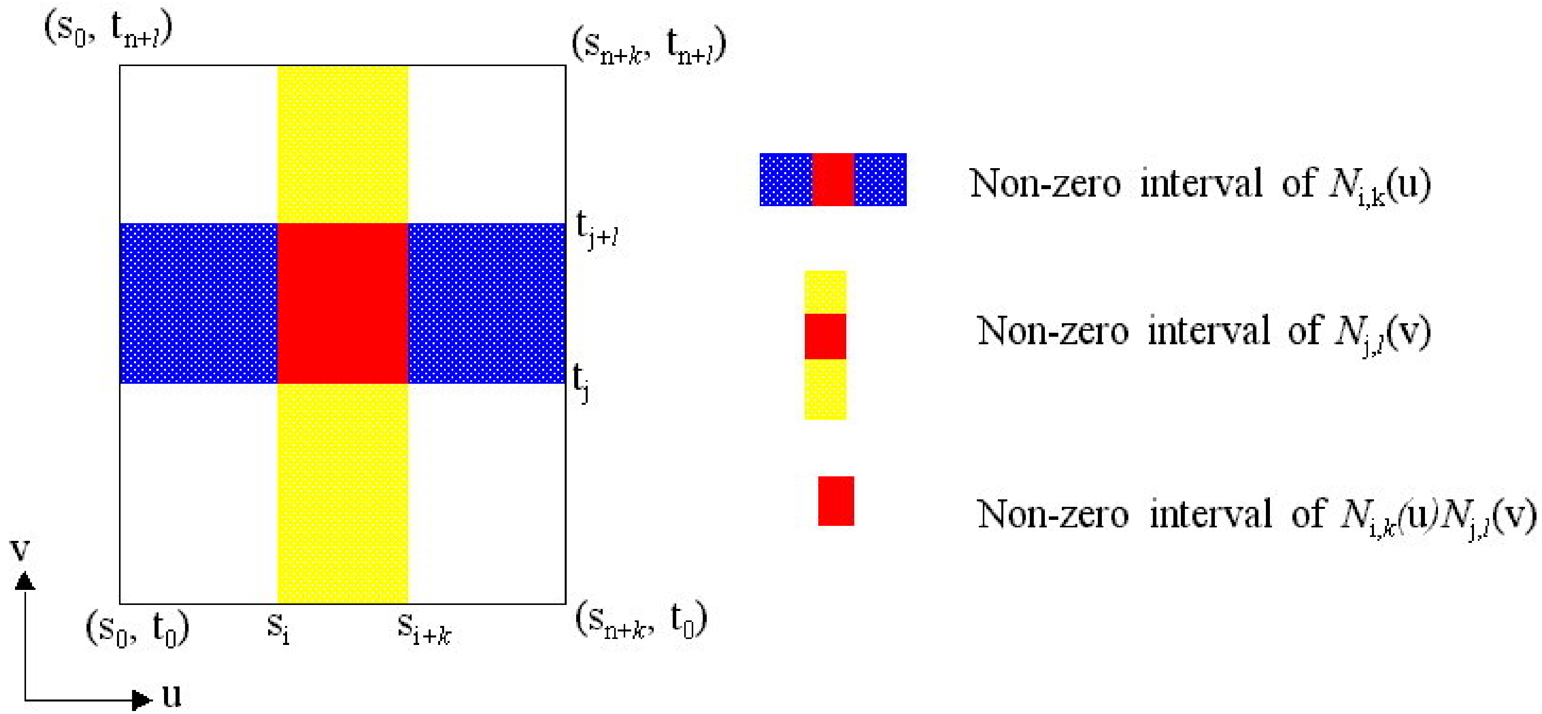
- The partial derivative $\frac{\partial P(u,v)}{\partial u}\Big|_{(0,0)}$ is parallel to $(\mathbf{P}_{10} - \mathbf{P}_{00})$, and the partial derivative $\frac{\partial P(u,v)}{\partial v}\Big|_{(0,0)}$ is parallel to $(\mathbf{P}_{01} - \mathbf{P}_{00})$.

  (Similar arguments hold for the other three corners.)

- The partial derivatives $\frac{\partial P(u,v)}{\partial u}$ and $\frac{\partial P(u,v)}{\partial v}$ are also B-spline surfaces themselves.

- Local modification: when a $\mathbf{P}_{ij}$ is moved, only a local portion on the surface corresponding to some parameter subset $[u', u''] \times [v', v'']$ will be affected.
(http://madmax.me.berkeley.edu/~fuchung/Bsurface.html)

Local-modification Control of B-Spline Surface



$N_{i,k}(u)N_{j,l}(v)$ is the blending function before control point $\mathbf{P}_{ij}$. So when $\mathbf{P}_{ij}$ moves, it will only effect the portion of the shape of the original B-spline surface corresponding to the non-zero interval .

Example Questions

1. A conical surface is generated by revolving a straight line between two points (2, 0, 0) and (1, 2, 0) about the y-axis by 180 degrees.

   a. Derive the parametric equation of the conical surface. Assume that parameter u moves a points along a circle on a plane perpendicular to the y-axis as it varies from 0 to 1 and that parameter v changes the height of the circle in the y-direction as it varies from 0 to 1.

   b. Approximate the conical surface by a bicubic patch. In other words, you need to give all the 16 geometric coefficients of the bicubic patch.

   c. Evaluate the bicubic patch at u = 0.5 and v = 0.5 and compare the result with those from the exact parametric surface.

2. Refer to the figure below, it shows two semi-circles, in the xy-plane and xz-plane respectively. You are asked to derive two parametric surfaces based on these two semi-circles.

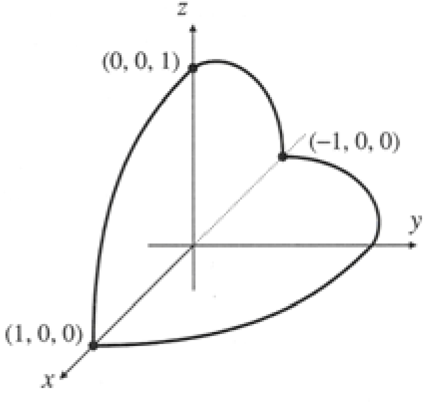
Example Questions (2)

2. (cont'd)

a. Define a parametric surface $P_1(u, v)$ by revolving the semi-circle in the xy-plane about the x-axis by 90 degrees, in which u is the parameter of the semi-circle from 0 to 1 and v (also from 0 to 1) is the parameter representing the revolving about the x-axis.

b. Define a Coons patch $P_2(u, v)$ based on the four arcs of the two semi-circles, i.e., the arc between (1, 0, 0) and (0, 0, 1) on the semi-circle in the xz-plane, etc. Make sure when constructing your Coons patch, the order of the four arcs and their parameter directions are correct.

c. Evaluate both $P_1(0.5, 0.5)$ and $P_2(0.5, 0.5)$. Are they equal to each other?



Example Questions (3)

3. Let $C_1(u)$ be a cubic Bezier curve with four control points P0, P1, P2, and P3, and $C_2(u)$ be another cubic Bezier curve with control points Q0, Q1, Q2, and Q3. Let S(u, v) be the ruled surface defined by the two rails $C_1(u)$ and $C_2(u)$.

   a. Give the parametric equation of S(u, v).

   b. Is S(u, v) a Bezier surface? If “Yes”, please represent S(u,v) in the standard Bezier surface representation. If “No”, prove it.

4. Continuing from Problem 3, suppose this time $C_1(u)$ remains the same, but $C_2(u)$ is now a degree-4 Bezier curve with five control points Q0, Q1, Q2, and Q4. Is S(u,v) a Bezier curve? Why?