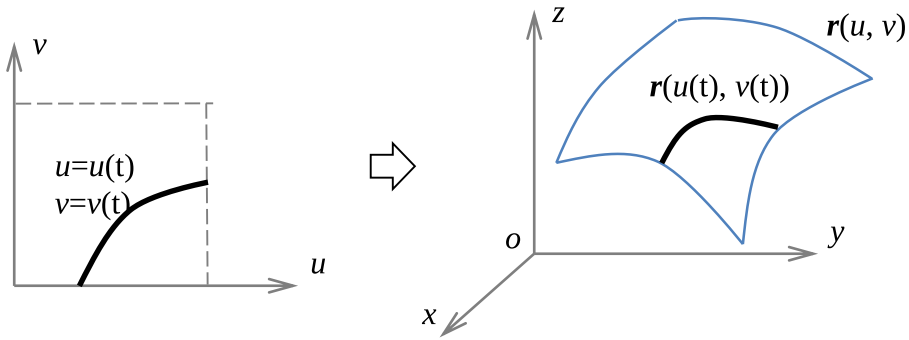
L8 – Differential Geometry of Surfaces

- We will discuss intrinsic properties of surfaces independent its parameterization
- Topics to be covered including:
  - Tangent plane and surface normal
  - First fundamental form I (metric)
  - Second fundamental form II (curvature)
  - Principal curvatures
  - Gaussian and mean curvatures
  - Euler's theorem

Tangent Plane and Surface Normal

- Consider a curve $u=u(\mathrm{t})$ $v=v(\mathrm{t})$ in the parametric domain of a parametric surface $\boldsymbol{r}=\boldsymbol{r}(u, v)$, then $\boldsymbol{r}=\boldsymbol{r}(t)=\boldsymbol{r}(u(\mathrm{t}), v(\mathrm{t}))$ is a parametric curve lying on the surface $\boldsymbol{r}$.

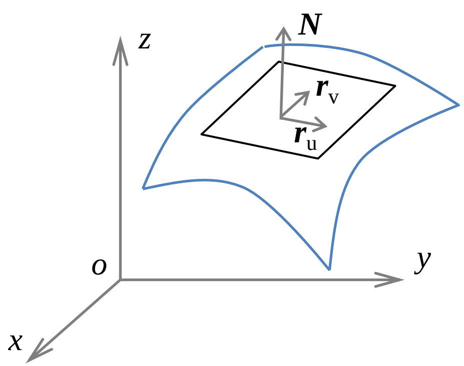


Using the chain rule on

$\boldsymbol{r}(t)=\boldsymbol{r}(u(\mathrm{t}), v(\mathrm{t}))$

$\Rightarrow$ $\dot{\boldsymbol{r}}(t) = \boldsymbol{r}_u \dot{u} + \boldsymbol{r}_v \dot{v}$

- The tangent plane at point $\boldsymbol{p}$ can be considered as a union of the tangent vectors of the form in above equation for all $\boldsymbol{r}(\mathrm{t})$ (i.e., many curves) passing through $\boldsymbol{p}$.



$$\boldsymbol{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}$$

Tangent Plane and Surface Normal (cont.)

- The equation of the tangent plane at $\boldsymbol{r}(u_\mathrm{p}, v_\mathrm{p})$ is

$$(\boldsymbol{p} - \boldsymbol{r}(u_\mathrm{p}, v_\mathrm{p})) \cdot \boldsymbol{N}(u_\mathrm{p}, v_\mathrm{p}) = 0$$

- **Definition:** A regular (ordinary) point p on a parametric surface is defined as a point where $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$. A point with $\boldsymbol{r}_u \times \boldsymbol{r}_v = 0$ is called a singular point.

- Implicit surface $f(x,y,z)=0$, the unit normal vector is given by

$$\boldsymbol{N} = \frac{\nabla f}{||\nabla f||}$$

First Fundamental Form I (Metric)

- If we define the parametric curve by a curve $u=u(\mathrm{t})$ $v=v(\mathrm{t})$ on a parametric surface $\boldsymbol{r}=\boldsymbol{r}(u,v)$

$$ds = \left\|\frac{d\boldsymbol{r}}{dt}\right\| dt = \left\|\boldsymbol{r}_u \frac{du}{dt} + \boldsymbol{r}_v \frac{dv}{dt}\right\| dt$$

$$= \sqrt{(\boldsymbol{r}_u \dot{u} + \boldsymbol{r}_v \dot{v}) \cdot (\boldsymbol{r}_u \dot{u} + \boldsymbol{r}_v \dot{v})}dt$$

$$= \sqrt{Edu^2 + 2Fdudv + Gdv^2} \quad (\because \dot{u} = \frac{du}{dt},\ \dot{v} = \frac{dv}{dt})$$

where

$$E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u,\ F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v, G = \boldsymbol{r}_v \cdot \boldsymbol{r}_v$$

- E, F, G are call the first fundamental form coefficients and play important role in many intrinsic properties of a surface

First Fundamental Form I (cont.)

- **Definition:** The first fundamental form is defined as

$$I = (ds)^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = Edu^2 + 2Fdudv + Gdv^2$$

which can also be rewritten as

$$I = \frac{1}{E}(Edu + Fdv)^2 + \frac{EG - F^2}{E}(dv)^2$$

$(a \times b) \cdot (c \times d)$
$=(a \cdot c)(b \cdot d)-(a \cdot d)(b \cdot c)$

- We can then have

$$E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u > 0$$

$$(\boldsymbol{r}_u \times \boldsymbol{r}_v)^2 = (\boldsymbol{r}_u \times \boldsymbol{r}_v) \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v)$$
$$= (\boldsymbol{r}_u \cdot \boldsymbol{r}_u)(\boldsymbol{r}_v \cdot \boldsymbol{r}_v) - (\boldsymbol{r}_u \cdot \boldsymbol{r}_v)^2 = EG - F^2 > 0$$

- Conclusion: we know that I is positive definite, provided that the surface is regular (since $I \geq 0$ and $I = 0$ if and only if $\mathrm{d}u=0$ and $\mathrm{d}v=0$ at the same time)

Example I: Let's compute the arc length of a curve $u = t$, $v = t$ for $0 \le t \le 1$ on a hyperbolic paraboloid $\boldsymbol{r}(u, v)=(u, v, uv)^{\mathrm{T}}$ where $0 \le u, v \le 1$.

Solution: $$\boldsymbol{r}_u = (1,0,v)^T, \boldsymbol{r}_v = (0,1,u)^T$$

$$E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u = 1 + v^2, G = \boldsymbol{r}_v \cdot \boldsymbol{r}_v = 1 + u^2, F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v = uv.$$

and along the curve the first fundamental form coefficients are

$$E = 1 + t^2, G = 1 + t^2, F = t^2.$$

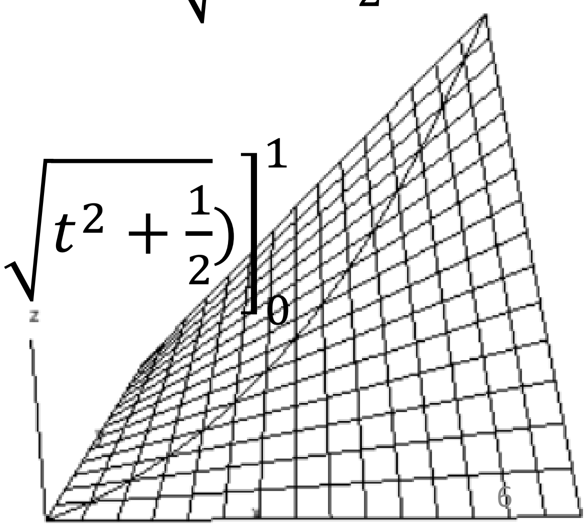
Thus

$$ds = \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}dt = \sqrt{2 + 4t^2}dt = 2\sqrt{t^2 + \frac{1}{2}}dt$$

The arc length

$$s = \int_0^1 2\sqrt{t^2 + \frac{1}{2}}dt = \left[t\sqrt{t^2 + \frac{1}{2}} + \frac{1}{2}\log(t + \sqrt{t^2 + \frac{1}{2}})\right]_0^1$$

$$= \sqrt{\frac{3}{2}} + \frac{1}{2}\log(\sqrt{2} + \sqrt{3})$$



First Fundamental Form I (Angle)

- The angle between two curves on a parametric surface

$$\boldsymbol{r}_1 = \boldsymbol{r}(u_1(t), v_1(t)), \boldsymbol{r}_2 = \boldsymbol{r}(u_2(t), v_2(t))$$

can be evaluated by tacking the inner product of their tangent vectors.

$$\cos\omega = \frac{d\boldsymbol{r}_1 \cdot \mathrm{d}\boldsymbol{r}_2}{\|d\boldsymbol{r}_1\|\|d\boldsymbol{r}_2\|}$$

$$= \frac{Edu_1du_2 + F(du_1dv_2 + dv_1du_2) + Gdv_1dv_2}{\sqrt{Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2}\sqrt{Edu_2^2 + 2Fdu_2dv_2 + Gdv_2^2}}$$

- As a result, the orthogonality condition for $\dot{\boldsymbol{r}}_1$ and $\dot{\boldsymbol{r}}_2$ is

$$Edu_1du_2 + F(du_1dv_2 + dv_1du_2) + Gdv_1dv_2 = 0$$

- In particular, the angle of two iso-parametric curves is

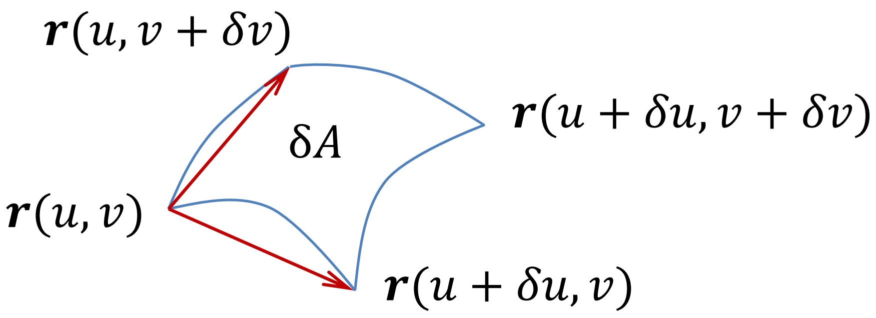
$$\cos\omega = \frac{\boldsymbol{r}_u \cdot \boldsymbol{r}_v}{\|\boldsymbol{r}_u\|\|\boldsymbol{r}_v\|} = \frac{\boldsymbol{r}_u \cdot \boldsymbol{r}_v}{\sqrt{\boldsymbol{r}_u \cdot \boldsymbol{r}_u}\sqrt{\boldsymbol{r}_v \cdot \boldsymbol{r}_v}} = \frac{F}{\sqrt{EG}}$$

Thus, the iso-parametric curves are orthogonal if $F \equiv 0$.

First Fundamental Form I (Area)

- The area of a small parallelogram with 4 vertices

$$\boldsymbol{r}(u,v), \boldsymbol{r}(u+\delta u, v), \boldsymbol{r}(u, v+\delta v), \boldsymbol{r}(u+\delta u, v+\delta v)$$



is approximated by

$$\delta A = \|\boldsymbol{r}_u \delta u \times \boldsymbol{r}_v \delta v\| = \sqrt{EG - F^2}\,\delta u \delta v$$

$$(\because (\boldsymbol{r}_u \times \boldsymbol{r}_v)^2 = (\boldsymbol{r}_u \cdot \boldsymbol{r}_u)(\boldsymbol{r}_v \cdot \boldsymbol{r}_v) - (\boldsymbol{r}_u \cdot \boldsymbol{r}_v)^2)$$

Or in different form as

$$dA = \sqrt{EG - F^2}\,du dv$$

Example II: Let's compute the area of a region on the hyperbolic paraboloid. The region is bounded by positive u and v axes and a quarter circle $u^2 + v^2 = 1$ as shown.

Solution: Again

$$\boldsymbol{r}_u = (1,0,v)^T, \boldsymbol{r}_v = (0,1,u)^T$$

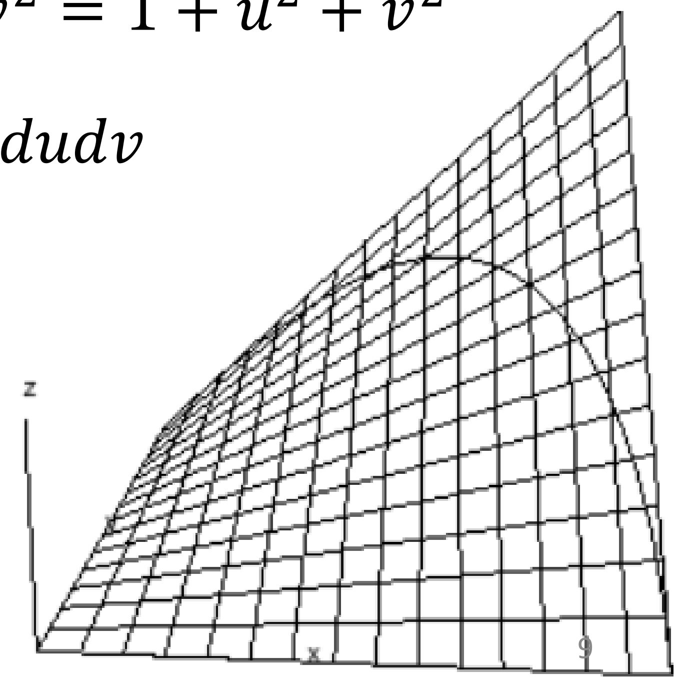
$$E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u = 1 + v^2, G = \boldsymbol{r}_v \cdot \boldsymbol{r}_v = 1 + u^2, F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v = uv$$

$$EG - F^2 = (1 + v^2)(1 + u^2) - u^2v^2 = 1 + u^2 + v^2$$

$$\therefore A = \int_D \sqrt{1 + u^2 + v^2} dudv$$

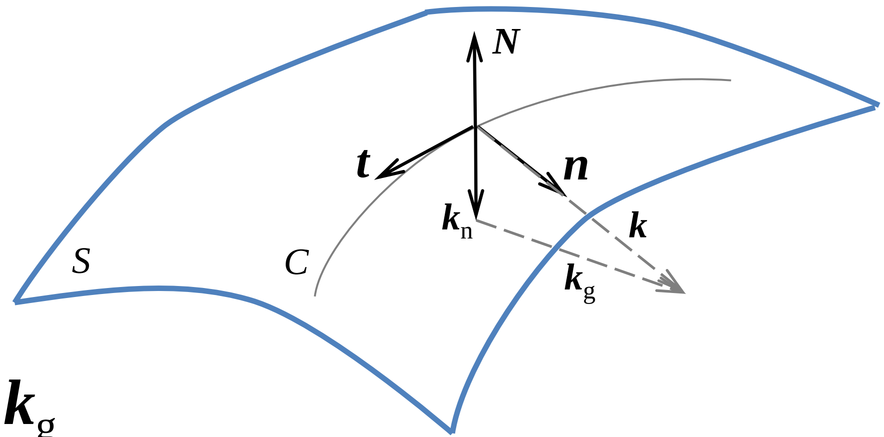
Letting $u = r\cos\theta$ and $v = r\sin\theta$

$$A = \int_0^{\frac{\pi}{2}} \int_0^1 r\sqrt{1 + r^2}\, drd\theta = \frac{\pi}{6}(\sqrt{8} - 1)$$



Second Fundamental Form II (Curvature)

- To quantify the curvature of a surface S, we consider a curve C on S which pass through the point $\boldsymbol{p}$.
- The unit tangent vector $\boldsymbol{t}$ and



the unit normal vector $\boldsymbol{n}$ of the curve C are related by

$$\boldsymbol{k} = \mathrm{d}\boldsymbol{t}/\mathrm{ds} = k\mathbf{n} = \boldsymbol{k}_{\mathrm{n}} + \boldsymbol{k}_{\mathrm{g}}$$

where $\boldsymbol{k}_{\mathrm{n}}$ is the normal curvature vector and $\boldsymbol{k}_{\mathrm{g}}$ is the geodesic curvature vector.

$\boldsymbol{k}_{\mathrm{g}}$: the component of $\boldsymbol{k}$ in the direction perpendicular to $\boldsymbol{t}$ in the surface tangent plane

$\boldsymbol{k}_{\mathrm{n}}$: the component of $\boldsymbol{k}$ in the surface normal direction

$\boldsymbol{k}_{\mathrm{n}} = k_{\mathrm{n}}\boldsymbol{N}$, where k_{n} - the normal curvature of surface at $\boldsymbol{p}$ in $\boldsymbol{t}$ direction.

Second Fundamental Form II (cont.)

By differentiating $\boldsymbol{N} \cdot \boldsymbol{t} = 0$ along the curve with respect to *s*, we have

$$\frac{d\boldsymbol{t}}{ds} \cdot \boldsymbol{N} + \boldsymbol{t} \cdot \frac{d\boldsymbol{N}}{ds} = 0$$

Thus

$$k_n = \frac{d\boldsymbol{t}}{ds} \cdot \boldsymbol{N} = -\boldsymbol{t} \cdot \frac{d\boldsymbol{N}}{ds} = -\frac{d\boldsymbol{r}}{ds} \cdot \frac{d\boldsymbol{N}}{ds}$$

$$= \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}$$

$$\because d\boldsymbol{r} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv$$
$$d\boldsymbol{N} = \boldsymbol{N}_u du + \boldsymbol{N}_v dv$$
$$\therefore d\boldsymbol{r} \cdot d\boldsymbol{N} = (\boldsymbol{r}_u \cdot \boldsymbol{N}_u) du^2$$
$$+ (\boldsymbol{r}_u \cdot \boldsymbol{N}_v + \boldsymbol{r}_v \cdot \boldsymbol{N}_u) du dv$$
$$+ (\boldsymbol{r}_v \cdot \boldsymbol{N}_v) dv^2$$

with

$$L = -\boldsymbol{r}_u \cdot \boldsymbol{N}_u, \qquad N = -\boldsymbol{r}_v \cdot \boldsymbol{N}_v,$$

$$M = -\frac{1}{2}(\boldsymbol{r}_u \cdot \boldsymbol{N}_v + \boldsymbol{r}_v \cdot \boldsymbol{N}_u) = -\boldsymbol{r}_u \cdot \boldsymbol{N}_v = -\boldsymbol{r}_v \cdot \boldsymbol{N}_u$$

Second Fundamental Form II (cont.)

- Since $\boldsymbol{r}_{\mathrm{u}}$ and $\boldsymbol{r}_{\mathrm{v}}$ are perpendicular to $\boldsymbol{N}$, we have

$$\boldsymbol{r}_u \cdot \boldsymbol{N} = 0 \quad \Longrightarrow \quad \boldsymbol{r}_{uu} \cdot \boldsymbol{N} + \boldsymbol{r}_u \cdot \boldsymbol{N}_u = 0 \quad \text{(for } L\text{)}$$

$$\boldsymbol{r}_v \cdot \boldsymbol{N} = 0 \quad \Longrightarrow \quad \boldsymbol{r}_v \cdot \boldsymbol{N}_u + \boldsymbol{r}_{uv} \cdot \boldsymbol{N} = 0 \quad \text{(for } M\text{)}$$

$$\boldsymbol{r}_v \cdot \boldsymbol{N} = 0 \quad \Longrightarrow \quad \boldsymbol{r}_{vv} \cdot \boldsymbol{N} + \boldsymbol{r}_v \cdot \boldsymbol{N}_v = 0 \quad \text{(for } N\text{)}$$

hence: $L = \boldsymbol{r}_{uu} \cdot \boldsymbol{N}$, $M = \boldsymbol{r}_{uv} \cdot \boldsymbol{N}$, $N = \boldsymbol{r}_{vv} \cdot \boldsymbol{N}$

- We define the second fundamental form II as

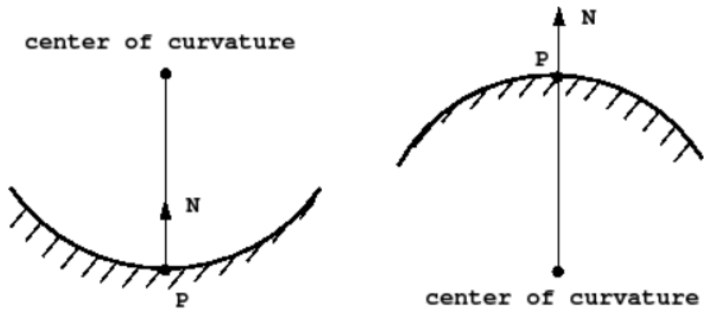
$$II = Ldu^2 + 2Mdudv + Ndv^2$$

and L, M, N are called second fundamental form coefficients

$$k_n = \frac{II}{I} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + NG^2}$$

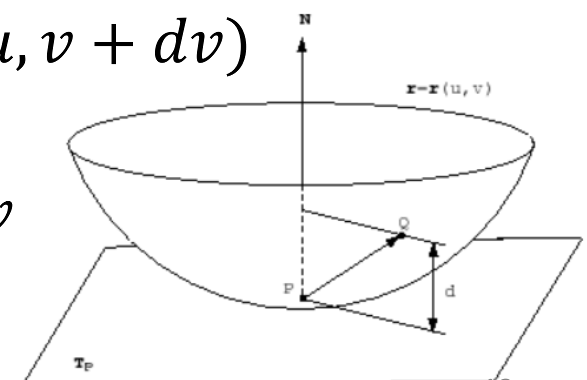
where $\lambda = \frac{dv}{du}$ is the direction of the tangent line to C at $\boldsymbol{p}$.

- **Theorem:** All curves lying on a surface S passing through a given point $\boldsymbol{p} \in S$ with the same tangent line have the same normal curvature at this point.
- Note that, sometimes the positive normal curvature is defined in the opposite direction.



- We now start to study the local surface property by II.
- Suppose two neighboring points $\boldsymbol{P}$ and $\boldsymbol{Q}$ on S

$$\boldsymbol{P} = \boldsymbol{r}(u, v) \quad \text{and} \quad \boldsymbol{Q} = \boldsymbol{r}(u + du, v + dv)$$



By Taylor expansion, we have

$$\boldsymbol{r}(u + du, v + dv) = \boldsymbol{r}(u, v) + \boldsymbol{r}_u du + \boldsymbol{r}_v dv$$
$$+ \frac{1}{2}(\boldsymbol{r}_{uu} du^2 + 2\boldsymbol{r}_{uv} du dv + \boldsymbol{r}_{vv} dv^2) + \cdots$$

- Projecting the vector:

$$\boldsymbol{PQ} = \boldsymbol{r}(u+du, v+dv) - \boldsymbol{r}(u,v)$$

$$= \boldsymbol{r}_u du + \boldsymbol{r}_v dv + \frac{1}{2}(\boldsymbol{r}_{uu} du^2 + 2\boldsymbol{r}_{uv} du dv + \boldsymbol{r}_{vv} dv^2) + \cdots$$

onto $\boldsymbol{N}$ by using the second fundamental form, we have

$$\boldsymbol{PQ} \cdot \boldsymbol{N} = (\boldsymbol{r}_u du + \boldsymbol{r}_v dv) \cdot \boldsymbol{N} + \frac{1}{2} II = \frac{1}{2} II$$

$(\because \boldsymbol{r}_u \cdot \boldsymbol{N} = \boldsymbol{r}_v \cdot \boldsymbol{N} = 0)$

where the higher order terms are neglected.

- In conclusion, the local shape variation:

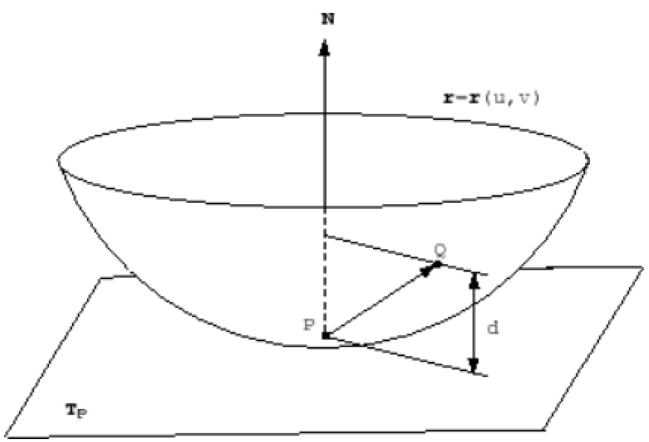
$$d = II / 2$$

When $d = 0$, it become

$$L du^2 + 2M du dv + N dv^2 = 0$$

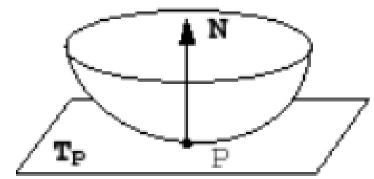
which can be considered as a quadratic equation in terms of du, dv.

$$du = \frac{-M \pm \sqrt{M^2 - LN}}{L} dv \qquad \text{(Assuming } L \neq 0\text{)}$$



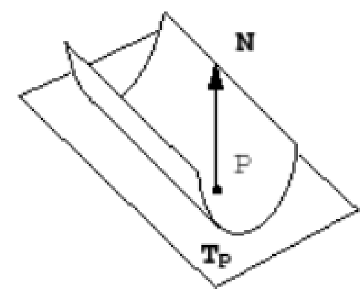
Local Shape Analysis by Second Fundamental Form II

- $M^2 - LN < 0$, this is no real root: No intersection between $\boldsymbol{r}(u,v)$ and T_p except the point $\boldsymbol{p}$.
  ⇨ An elliptic point



- $M^2 - LN = 0$ and $L^2 + M^2 + N^2 \neq 0$, there are double roots: The surface intersects T_p with one line

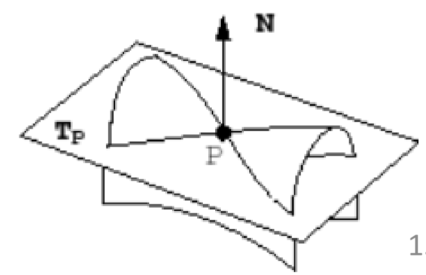
$$du = -\frac{M}{L}dv$$



  ⇨ A parabolic point

- $M^2 - LN > 0$, there are two roots: The surface intersects its tangent lane with two lines

$$du = \frac{-M \pm \sqrt{M^2 - LN}}{L}dv$$



  ⇨ A hyperbolic point

Local Shape Analysis by Second Fundamental Form II (cont.)

- $L = M = N = 0$, the surface and the tangent plane have a contact of higher order than in the above cases at the point $\boldsymbol{p}$.

  ⇨ A flat or planar point

- If $L = 0$ and $N \neq 0$, we can solve for dv instead of du.
- If $L = N = 0$ and $M \neq 0$, we have $2Mdudv = 0$, thus the intersection lines will be the two iso-parametric lines:

$$u = const1 \qquad v = const2$$

Principal Curvatures

- From the second fundamental form, we already know

$$k_n = \frac{II}{I} = \frac{L+2M\lambda+N\lambda^2}{E+2F\lambda+NG^2}$$

The normal curvature at ***p*** depends on the direction of $\lambda = \frac{dv}{du}$.

- The extreme value of k_n can be obtained by solving: $dk_n/d\lambda = 0$.

$$(E + 2F\lambda + G\lambda^2)(N\lambda + M) - (L + 2M\lambda + N\lambda^2)(G\lambda + F) = 0$$

$$k_n^e = \frac{L+2M\lambda+N\lambda^2}{E+2F\lambda+G\lambda^2} = \frac{N\lambda+M}{G\lambda+F}$$

- Furthermore, since $E + 2F\lambda + G\lambda^2 = (E + F\lambda) + \lambda(F + G\lambda)$ and $L + 2M\lambda + N\lambda^2 = (L + M\lambda) + \lambda(M + N\lambda)$, we have

$$\Rightarrow \big((E + F\lambda) + \lambda(F + G\lambda)\big)(N\lambda + M) = \big((L + M\lambda) + \lambda(M + N\lambda)\big)(G\lambda + F)$$

$$\Rightarrow (E + F\lambda)(N\lambda + M) = (L + M\lambda)(G\lambda + F)$$

- Therefore, we have

$$k_n^e = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{M + N\lambda}{F + G\lambda} = \frac{L + M\lambda}{E + F\lambda}$$

- Substituting $\lambda = \frac{dv}{du}$, we have

$$(L - k_n^e E)du + (M - k_n^e F)dv = 0$$
$$(M - k_n^e F)du + (N - k_n^e G)dv = 0$$

This is a homogeneous linear system of equations for du, dv, which will have a nontrivial solution if and only if

$$\begin{vmatrix} L - k_n^e E & M - k_n^e F \\ M - k_n^e F & N - k_n^e G \end{vmatrix} = 0$$

$$(EG - F^2){k_n^e}^2 - (EN + GL - 2FM)k_n^e + (LN - M^2) = 0$$

- The discriminant of D of this quadratic equation in k_n^e can be

$$D = 4\left(\frac{EG - F^2}{E^2}\right)(EM - FL)^2 + \left(EN - GL - \frac{2F}{E}(EM - FL)\right)^2$$

Since $EG - F^2 \geq 0$, we have $D \geq 0$. The quadratic equation has real roots.

- If we set

$$K = \frac{LN - M^2}{EG - F^2} \quad \text{and} \quad H = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

The quadratic equation is simplified to

$${k_n^e}^2 - 2Hk_n^e + K = 0$$

which has the following solutions

$$k_{\max} = H + \sqrt{H^2 - K}$$
$$k_{\min} = H - \sqrt{H^2 - K}$$

- $k_{\max}$ and $k_{\min}$ are defined as the maximum and minimum principal curvature.
- The directions for which give $k_{\max}$ and $k_{\min}$ are called principal directions, the corresponding directions in uv-plane

$$\lambda = -\frac{M - k_n F}{N - k_n G} \quad \text{or} \quad -\frac{L - k_n E}{M - k_n F}$$

by replacing k_n with either $k_{\max}$ or $k_{\min}$.

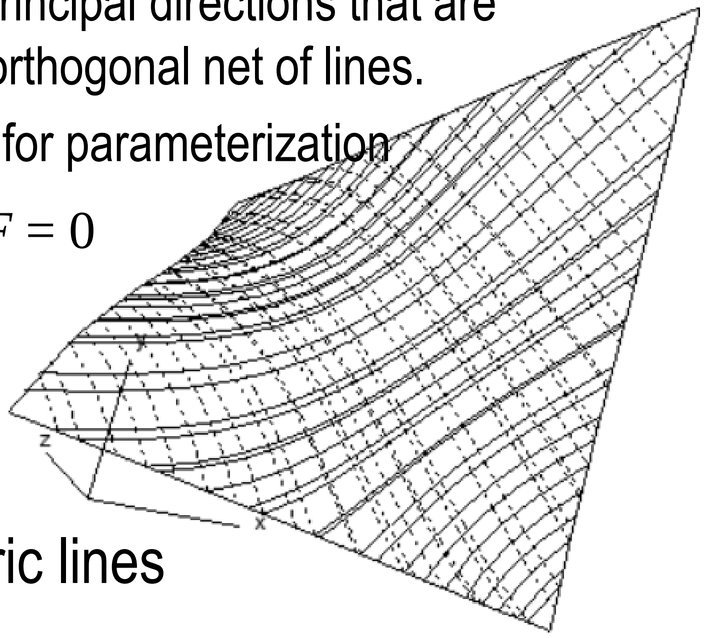
- When $H^2 = K$, k_n is a double root with value equal to H and the corresponding point of the surface is an umbilical point.
  - At an umbilical point, a surface is locally a part of sphere with radius of curvature: $1 / H$
  - In the special case – both K and H vanish, the point is a flat or planar point
- A curve on surface where tangent at each point is in a principal direction at that point is called a line of curvature
  - At each (non-umbilical) point, there are two principal directions that are orthogonal – the lines of curvatures form an orthogonal net of lines.
  - We can try to use this orthogonal net of lines for parameterization
  - Let $\cos\omega = \cos\frac{\pi}{2} = 0$ (page 7), we have $F = 0$
  - We can then from the equation of λ have

$$(L - k_n E)du + M dv = 0$$

$$M du + (N - k_n G)dv = 0$$

- The necessary condition for the parametric lines

to be line of curvature is: $F = M = 0$.



Line of curvatures

Gaussian and Mean Curvatures

- From above equation on page 19, we can easily have

$$K = k_{\max} k_{\min}$$
$$H = (k_{\min} + k_{\max})/2$$

- The sign of the Gaussian curvature coincides with the sign of $LN - M^2$, since $K = \frac{LN-M^2}{EG-F^2}$ and $EG - F^2 > 0$. Then, we have
  1. $K > 0$, the surface is elliptic at the point;
  2. $K < 0$, the surface is hyperbolic at the point;
  3. $K = 0$ and $H \neq 0$, the surface is parabolic at the point;
  4. $K = 0$ and $H = 0$, the surface is flat (or planar) at the point.
- This is good for segmenting an given surface to different regions (reverse engineering of CAD models).

http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.83.3315&rep=rep1&type=pdf

Euler's Theorem

- The normal curvature of a surface in an arbitrary direction (in the tangent plane) at point $\boldsymbol{p}$ can be expressed in terms of principal curvatures k_1 and k_2 at point $\boldsymbol{p}$ and the angle between the arbitrary direction and the principal direction corresponding to k_1 as:

$$k_n(\phi) = k_1 \cos^2 \phi + k_2 \sin^2 \phi$$