L5 – Implicit Surface Reconstruction

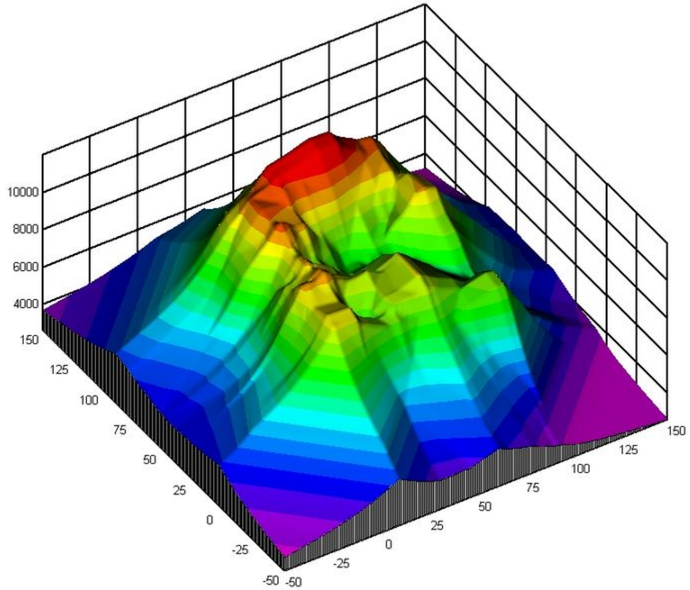
- Techniques to generate B-rep of surfaces with the help of implicit surfaces
  - Distance field
  - Radial Basis Function (RBF)
  - Multi-level Partition Unity (MPU) implicit
  - Poisson Reconstruction
  - Contouring methods for generating B-rep
    - Uniform sampling
    - Adaptive sampling

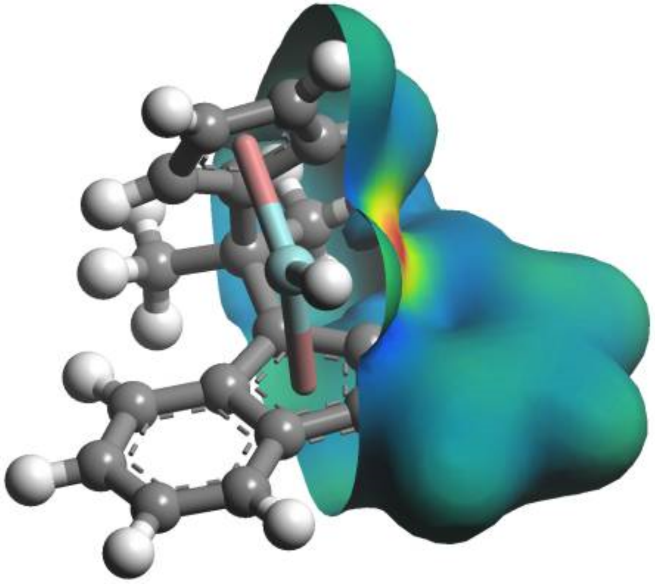
Implicit Functions & Implicit Surface

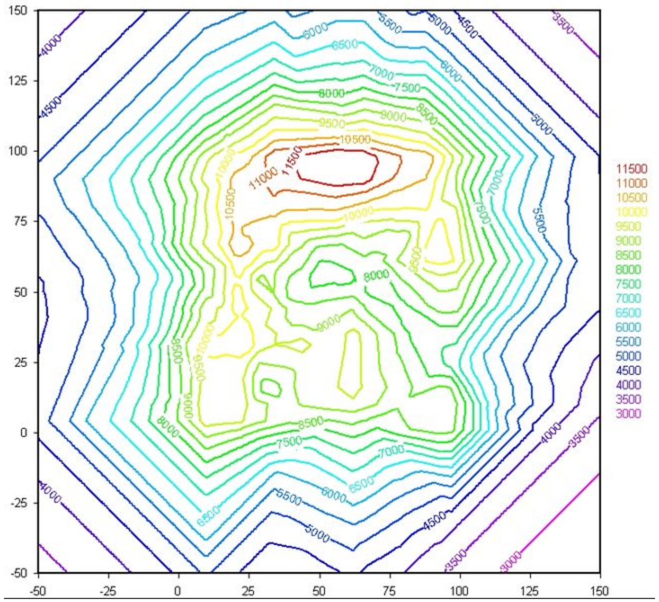
- In mathematics, an implicit function is a function in which the dependent variable has not been given "explicitly" in terms of the independent variable
- Implicit function based fitting (approximation or interpolation) is employed for surface reconstruction
- Advantages:
  - Compact mathematical representation
  - Easy topology change
  - Water-tight surface is always generated

Isoline and Isosurface

- An isoline of a function of two variables is a curve along which the function has a constant value
- An isosurface is a 3D analog







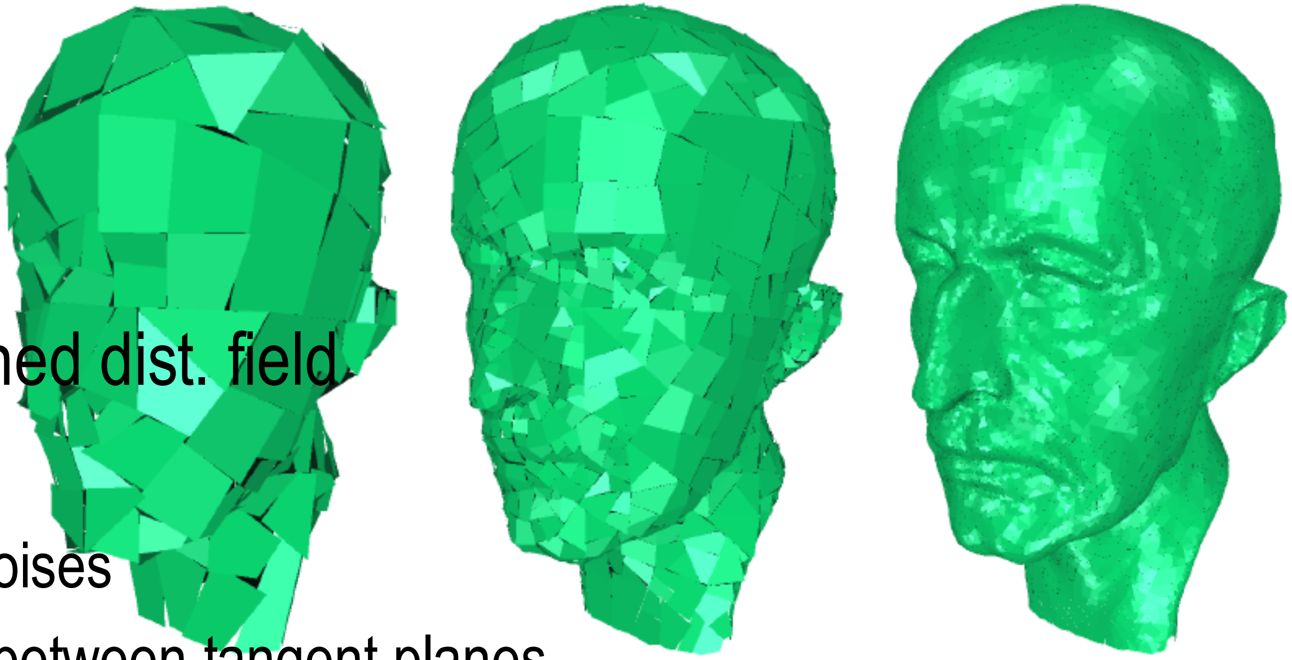
Distance Field Based Reconstruction

- Signed distance function: $f(\mathbf{p})$ from an arbitrary point **p** in 3D to a known surface *M* is the distance between **p** and the closest point **z** on *M* multiplied by ±1
  - Sign depending on which side of the surface **p** lies
  - Surface is defined at $f(\mathbf{p})=0$
- In reality *M* is not known, but we can mimic this procedure using the oriented tangent planes
  - First, find the tangent plane $T_p(\boldsymbol{x}_i)$ whose center $\mathbf{o}_i$ is closest to **p**
  - The signed distance function is approximated by

$$f(\mathrm{p}) = \mathrm{dist}_i(\mathrm{p}) = (\mathrm{p} - \mathrm{o}_i) \cdot \hat{\mathbf{n}}_i$$

the distance between **p** to its projection on the plane $T_p(\boldsymbol{x}_i)$

Dist. Field



- Piece-wise signed dist. field
- Main problems
  - Influence of noises
  - Compatibility between tangent planes

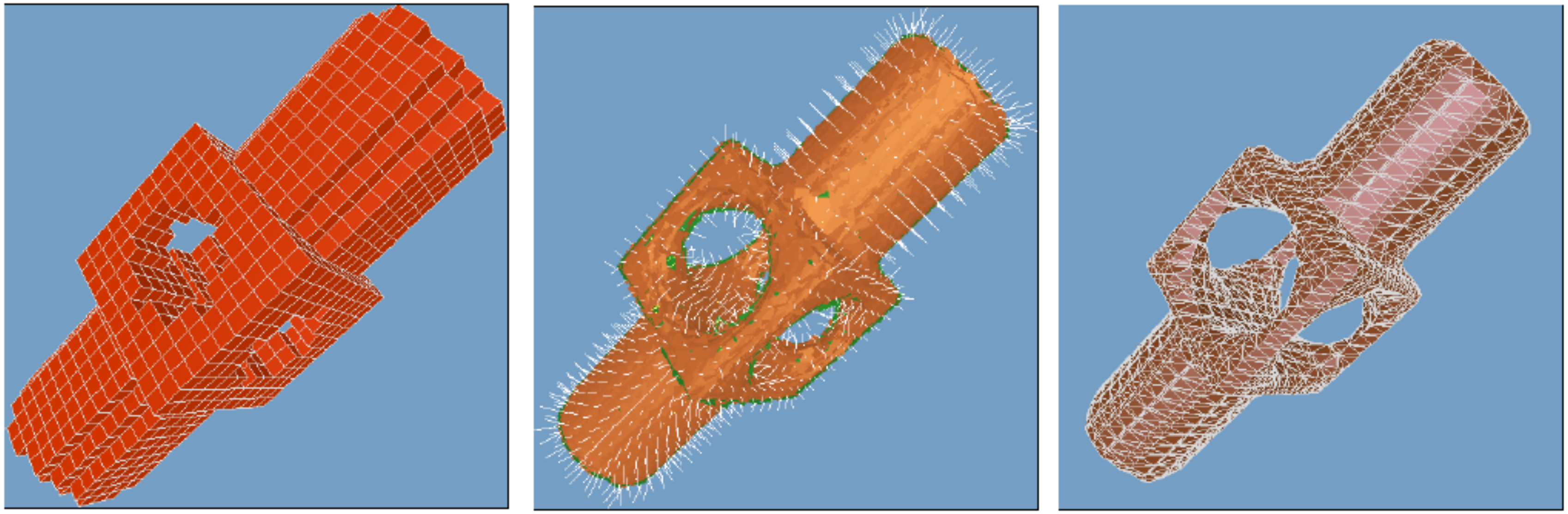
ρ-dense, δ-noisy sample

```
$i \leftarrow$ index of tangent plane whose center is closest to $\mathbf{p}$

{ Compute $\mathbf{z}$ as the projection of $\mathbf{p}$ onto $Tp(\mathbf{x}_i)$ }
$\mathbf{z} \leftarrow \mathbf{o}_i - ((\mathbf{p} - \mathbf{o}_i) \cdot \hat{\mathbf{n}}_i)\, \hat{\mathbf{n}}_i$

if $d(\mathbf{z}, X) < \rho + \delta$ then
    $f(\mathbf{p}) \leftarrow (\mathbf{p} - \mathbf{o}_i) \cdot \hat{\mathbf{n}}_i$        $\{= \pm \|\mathbf{p} - \mathbf{z}\|\}$
else
    $f(\mathbf{p}) \leftarrow$ undefined
endif
```

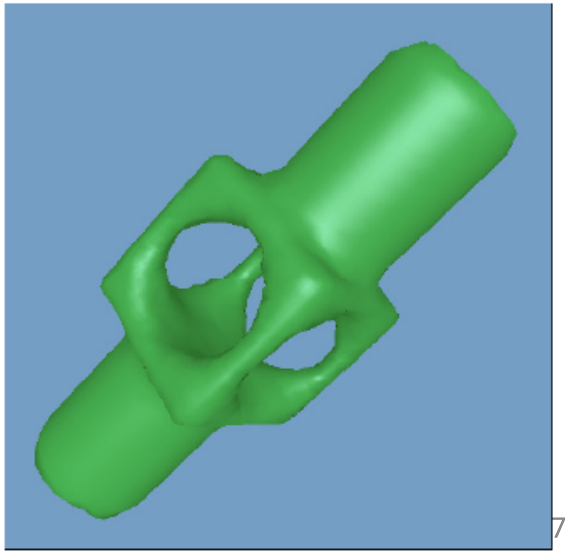
Mesh Generation from Distance Field

- A variation of Marching Cubes algorithm
  - To accurately estimate boundaries, the cube size should be set so that edges are of length less than δ+ρ
  - Signed distance function *f* is evaluated only at points close to data
  - No intersection is reported within a cube if the signed distance function is undefined at any vertex of the cube, thereby giving rise to boundaries in the simplicial surface
  - As a result of MC, triangles with poor shape will be generated
  - How about the intersection on edges?

Problems of Distance Based Method



- Relies too much on the quality of input points
- The compatibility between neighboring tangent planes may has problem
- Cannot guarantee to generate water-tight surface since there is some “undefined” region



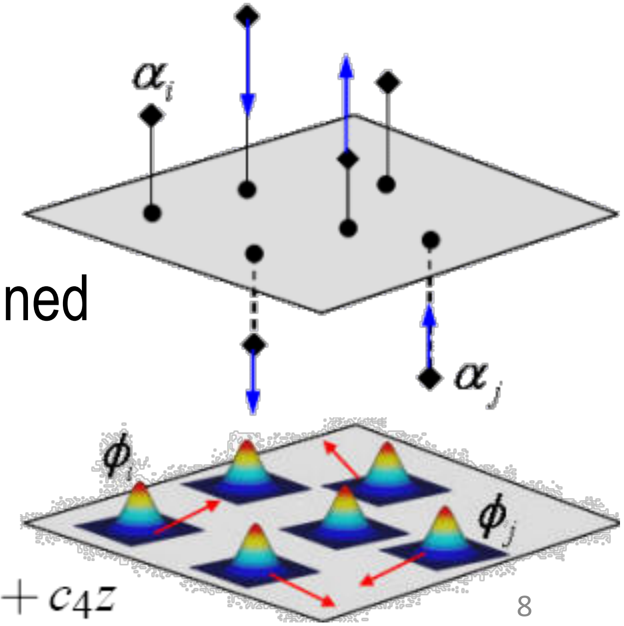
RBF Based Surface Reconstruction

- Fitting an implicit function to the given points

$$
\begin{aligned}
f(x_i, y_i, z_i) &= 0, & i &= 1, \ldots, n & &(\textit{on}\text{-surface points}),\\
f(x_i, y_i, z_i) &= d_i \neq 0, & i &= n+1, \ldots, N & &(\textit{off}\text{-surface points}).
\end{aligned}
$$

- Mathematically, a good choice - *radial basis function* (RBF)

$$s(x) = p(x) + \sum_{i=1}^{N} \lambda_i \phi(|x - x_i|)$$



- *p*(*x*) is a low-degree polynomial

- The basis function *Φ* is a real-value function defined on the interval [0,+∞)

2D: $\phi(r) = r^2 \log(r)$

3D: $\phi(r) = r^3$

$$p(x) = c_1 + c_2 x + c_3 y + c_4 z$$

RBF Based Surface Reconstruction

- To ensure that the obtained surface has integrable second derivatives, the following side condition must be added

$$\sum_{i=1}^{N} \lambda_i q(x_i) = 0, \text{ for all polynomials } q \text{ of degree at most } m.$$

lead to

e.g. $\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \lambda_i x_i = \sum_{i=1}^{N} \lambda_i y_i = \sum_{i=1}^{N} \lambda_i z_i = 0$

when using 1^{st} order polynomial in *p*(x)

$$\begin{pmatrix} A & P \\ P^\top & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ c \end{pmatrix} = B \begin{pmatrix} \lambda \\ c \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$$A_{i,j} = \phi(|x_i - x_j|), \qquad i,j = 1,\ldots,N,$$

$$P_{i,j} = p_j(x_i), \qquad i = 1,\ldots,N, \quad j = 1,\ldots,\ell.$$

*The matrix B typically has poor conditioning as the number of data points N gets larger, and it is a dense matrix

Solving RBF Based Reconstruction

- Direct solver does not work when *n*>2,000
- Fast Multi-pole Method (FMM) is employed
  - Fact: infinite precision is neither required nor expected
  - For the evaluation of an RBF, the approximations of choice are far- and near-field expansions
  - With the centers clustered in a hierarchical manner
  - far- and near-field expansions are used to generate an approximation to that part of the RBF due to the centers in a particular cluster
- Short Course at: www.math.nyu.edu/faculty/greengar/shortcourse_fmm.pdf

RBF Approximation of Noisy Data

- What if there are noises in data
- Consider this problem: $$\min_{s\in \mathrm{BL}^{(2)}(\mathbb{R}^3)} \rho\|s\|^2 + \frac{1}{N}\sum_{i=1}^{N}\left(s(x_i)-f_i\right)^2 \qquad \boxed{\rho \geq 0}$$

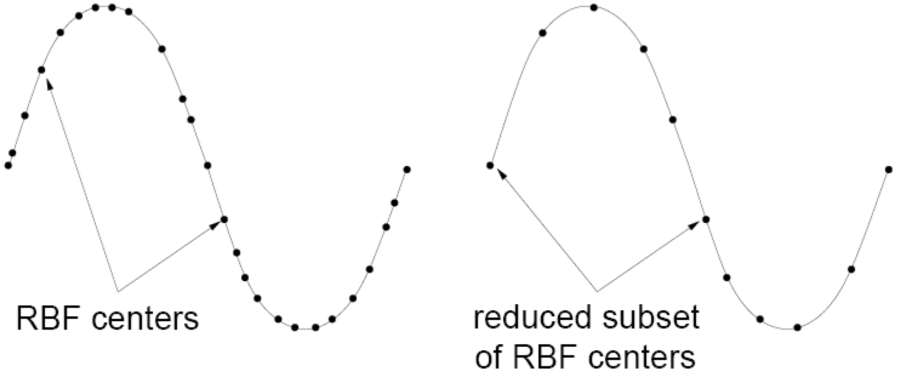
Regularization Term

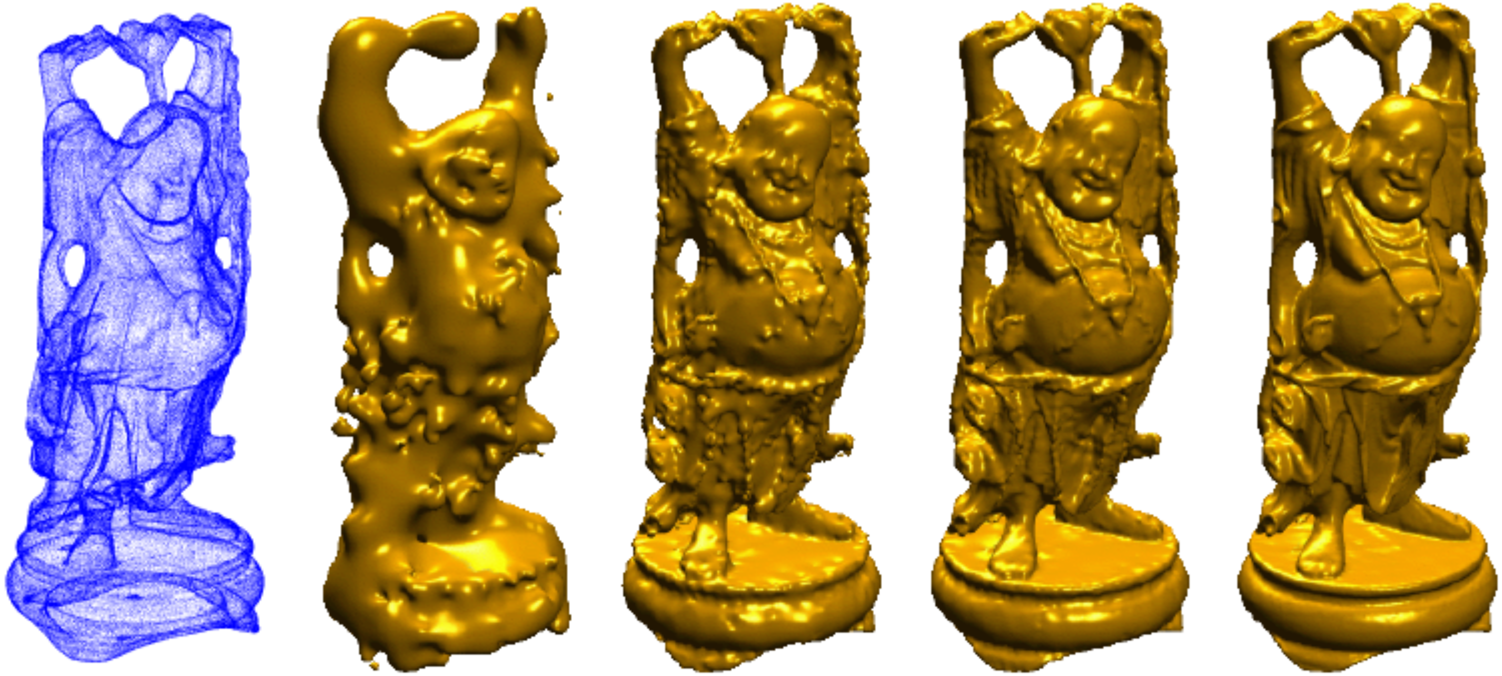
- Solution can be obtained by

$$\begin{pmatrix} A - 8N\pi\rho I & P \\ P^{\mathsf{T}} & 0 \end{pmatrix}\begin{pmatrix} \lambda \\ c \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

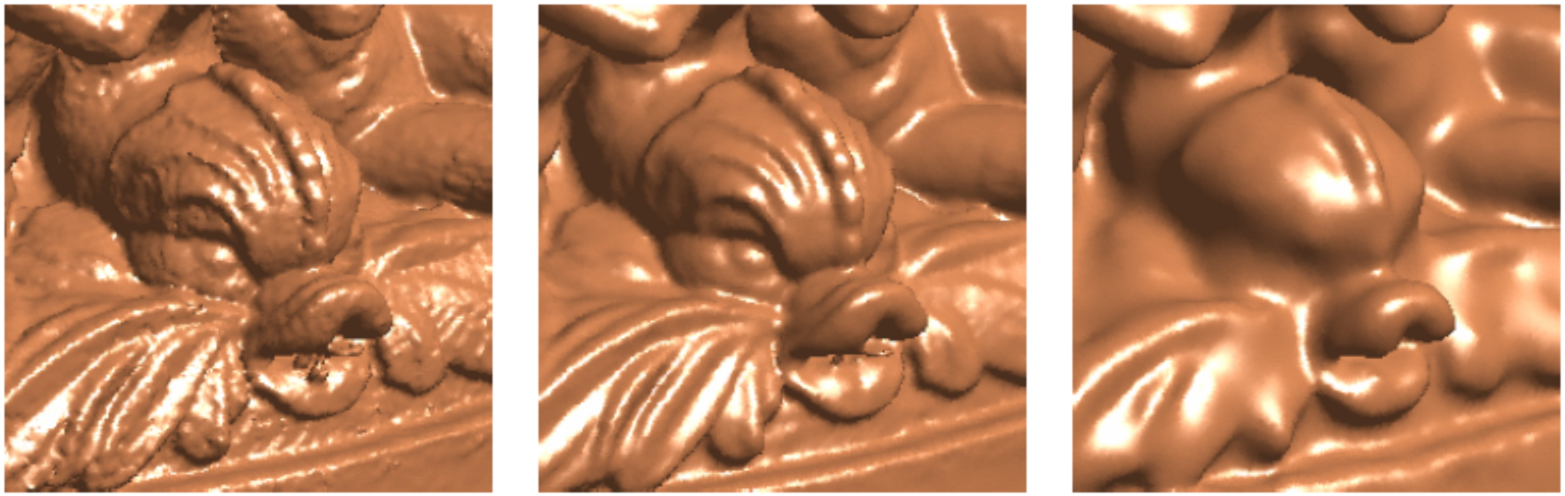
- Another method: greedy reduction

1. Choose a subset from the interpolation nodes x_i and fit an RBF only to these.
2. Evaluate the residual, $\varepsilon_i = f_i - s(x_i)$, at all nodes.
3. If $\max\{|\varepsilon_i|\} < \textit{fitting accuracy}$ then stop.
4. Else append new centers where ε_i is large.
5. Re-fit RBF and goto 2.





A greedy algorithm iteratively fits an RBF to a point cloud

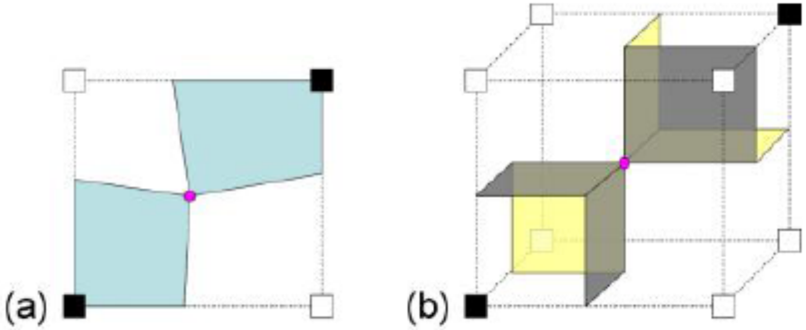


Exact fit medium amount of smoothing increased smoothing

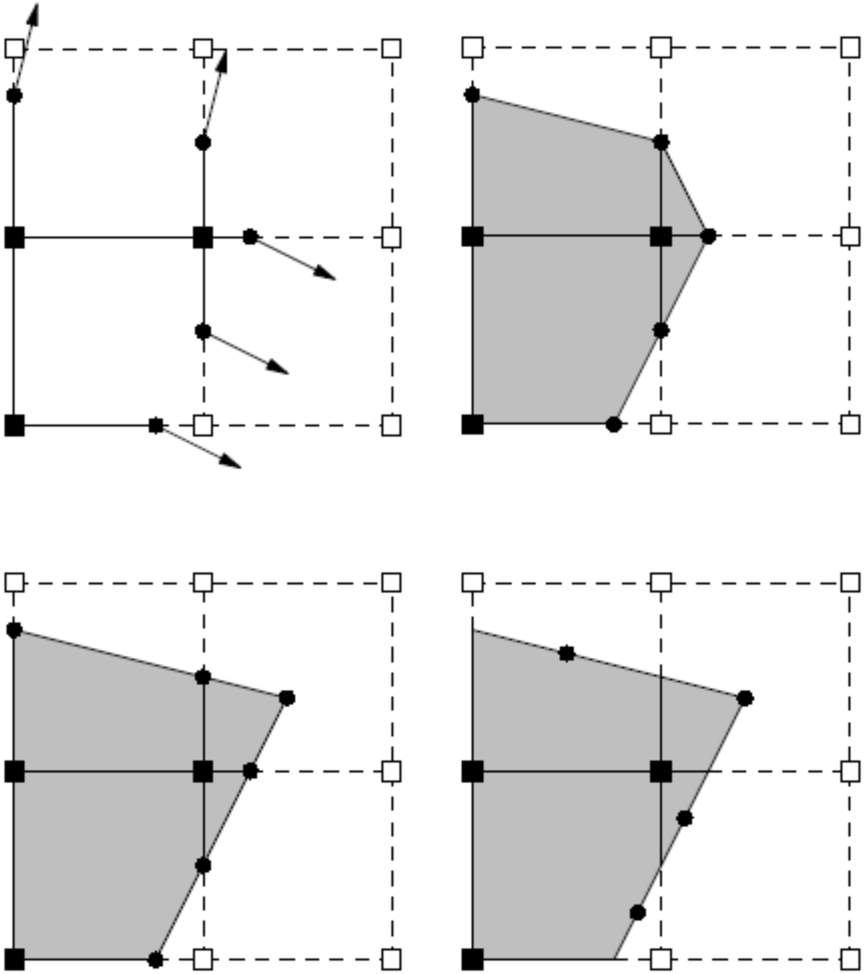
J. C. Carr, R. K. Beatson, J. B. Cherrie, T. J. Mitchell, W. R. Fright, B. C. McCallum, and T. R. Evans. *Reconstruction and representation of 3D objects with radial basis functions*. ACM SIGGRAPH '01

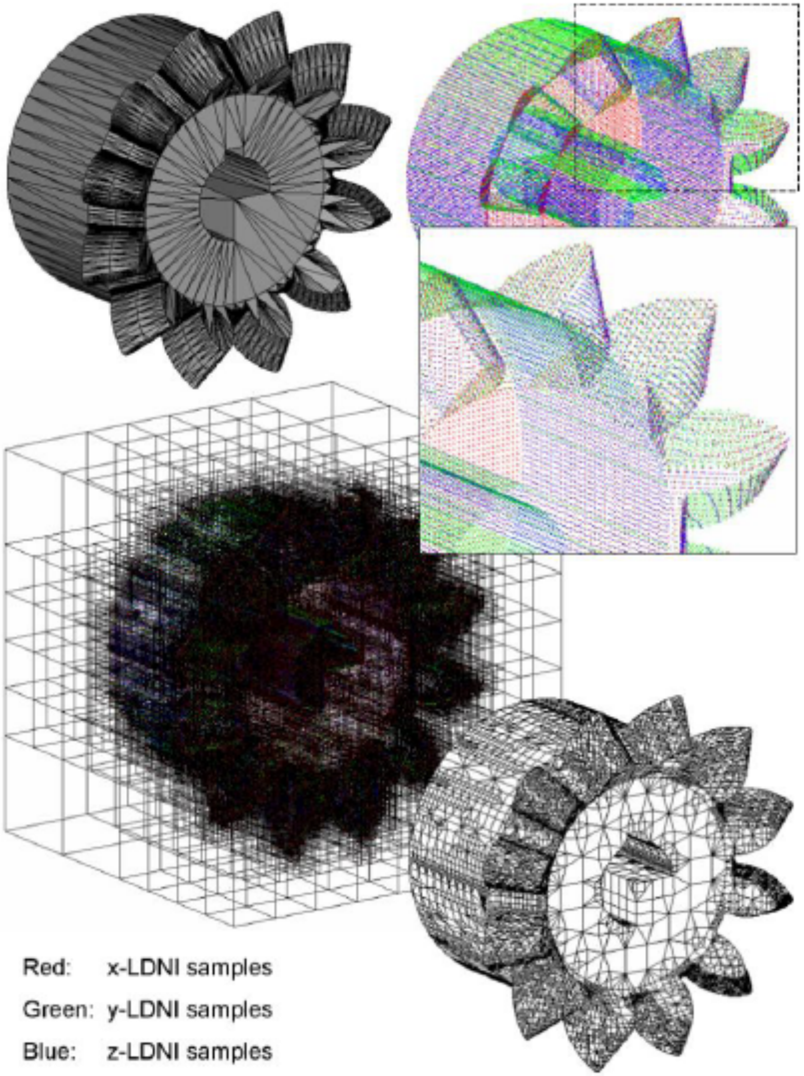
Mesh Generation from Implicit Surface

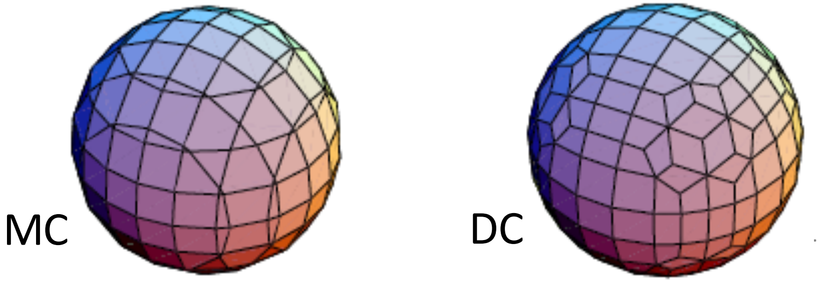
- Marching Cubes algorithm or its variants
  - Resolutions fixed
  - Does not adapt to the curvature of surface
  - Topology homeomorphism is not guaranteed
  - Sharp features are damaged
- Dual Contouring
  - Could be adaptive
  - Quadrangular/triangular mesh
  - Singular vertices can be generated
  - Sharp features can be reconstructed
  - Less number of polygons



MC vs. DC




Red: x-LDNI samples
Green: y-LDNI samples
Blue: z-LDNI samples


MC
DC

Partition of Unity

- Integrate locally defined approximants into a global approx.
- A bounded domain Ω in 3D and a set of nonnegative compactly supported functions: $\sum_i \varphi_i \equiv 1 \text{ on } \Omega$
- Let us associate a local approximation set of functions V_i with each sub-domain: $Q_i(\boldsymbol{x})$ (e.g., quadratic surface)
- Now an approximation of a function defined on Ω is

$$f(\mathbf{x}) \approx \sum_i \varphi_i(\mathbf{x}) Q_i(\mathbf{x}) \qquad Q_i \in V_i$$

- Given a set of nonnegative compactly supported functions $\{w_i\}$, we have that to $\Omega \subset \bigcup_i \operatorname{supp}(w_i)$, the $\{\varphi_i\}$ is defined by

$$\varphi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})}$$

Multi-level Partition of Unity Implicits

- For approximation purpose, using the quadratic B-spline function *b*(*t*) as

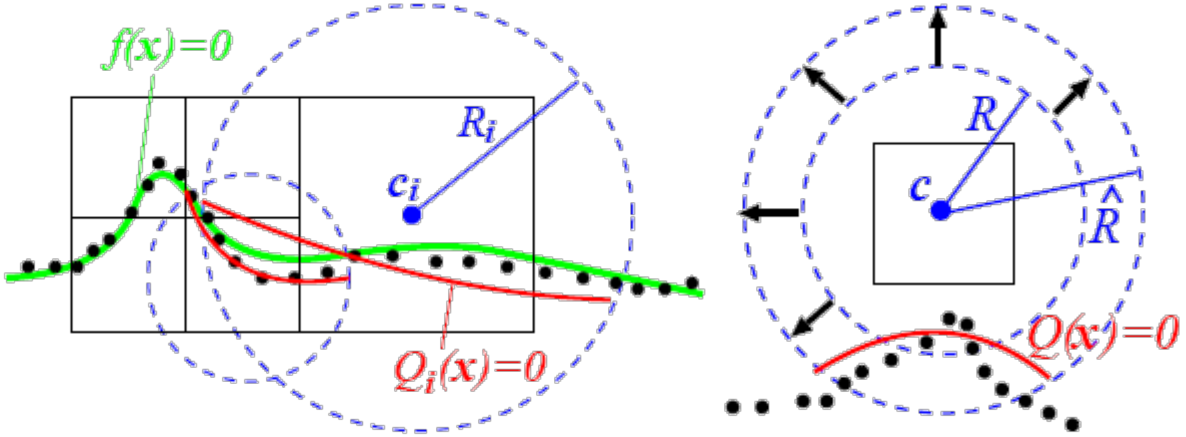
$$w_i(\mathbf{x}) = b\left(\frac{3\left|\mathbf{x}-\mathbf{c}_i\right|}{2R_i}\right)$$

  where $\mathbf{c}_i$ is the center, and R_i is the support size
- For interpolation purpose, using the inversed distance singular weights as

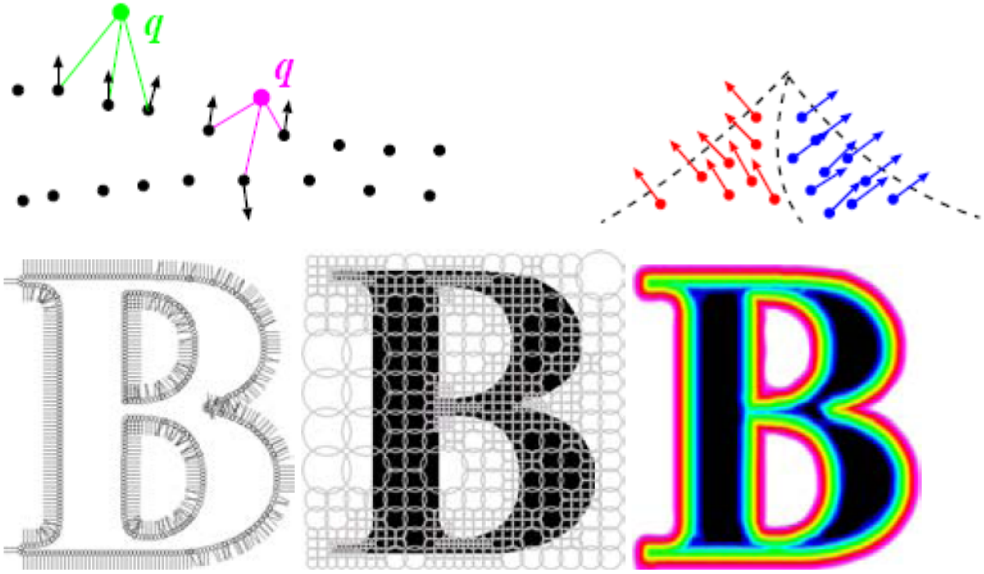
$$w_i(\mathbf{x}) = \left[\frac{\left(R_i - \left|\mathbf{x}-\mathbf{c}_i\right|\right)_+}{R_i\left|\mathbf{x}-\mathbf{c}_i\right|}\right]^2, \text{ where } (a)_+ = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{otherwise} \end{cases}$$

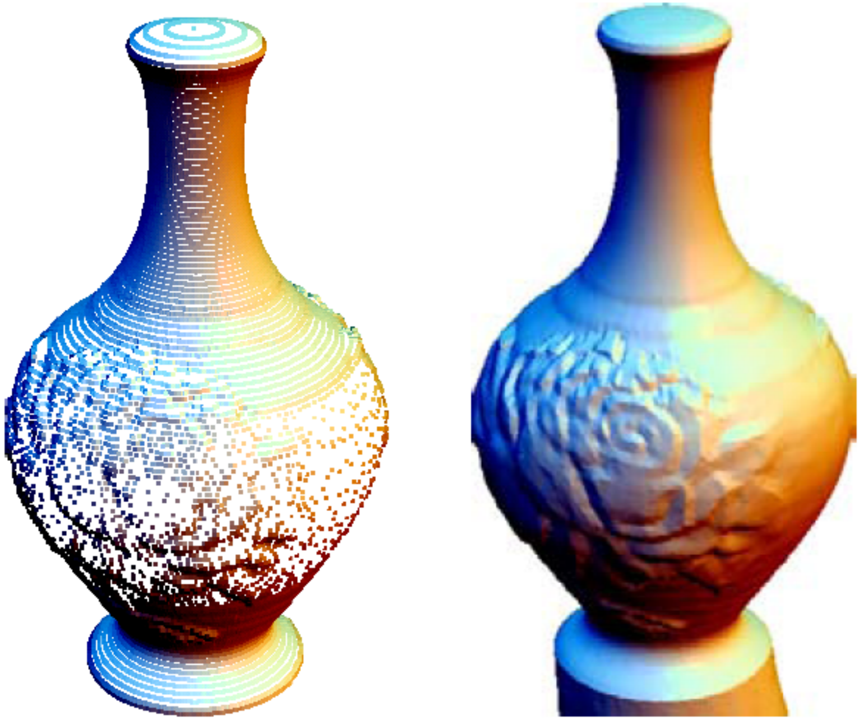
- Octree based approximate: using *R* as 0.75 of diagonal edge of each cell; or increase to enhance robustness

MPU Implicits



- When different situations fit different local implicits
- Better results can be given





Yutaka Ohtake, Alexander Belyaev, Marc Alexa, Greg Turk, and Hans-Peter Seidel. 2003. Multi-level partition of unity implicits. ACM Trans. Graph. 22, 3 (July 2003), 463-470.

Apply MPU to RBF

- First, construct the hierarchy of pnts: $\{\mathscr{P}^1, \mathscr{P}^2, \cdots, \mathscr{P}^M = \mathscr{P}\}$
- Then, starting from $f^0(\mathbf{x}) = -1$

$$f^k(\mathbf{x}) = f^{k-1}(\mathbf{x}) + o^k(\mathbf{x}) \qquad (k = 1, 2, \ldots, M),$$

where $f^k(\mathbf{x}) = 0$ interpolates $\mathscr{P}^k$. An offsetting function o^k

$$o^k(\mathbf{x}) = \sum_{\mathbf{p}_i^k \in \mathscr{P}^k} \left[g_i^k(\mathbf{x}) + \lambda_i^k \right] \phi_{\sigma^k}(\|\mathbf{x} - \mathbf{p}_i^k\|).$$

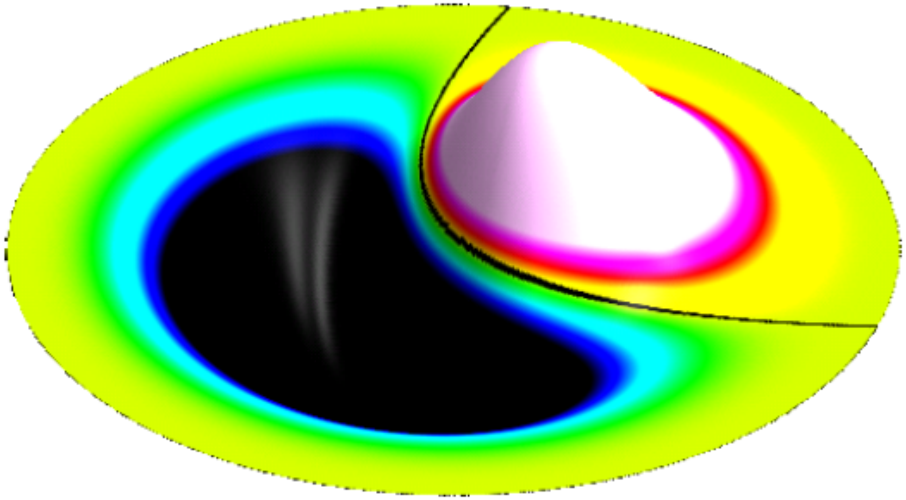
where local approximations $g_i^k(\mathbf{x})$ determined via least square fitting applied to $\mathscr{P}^k$

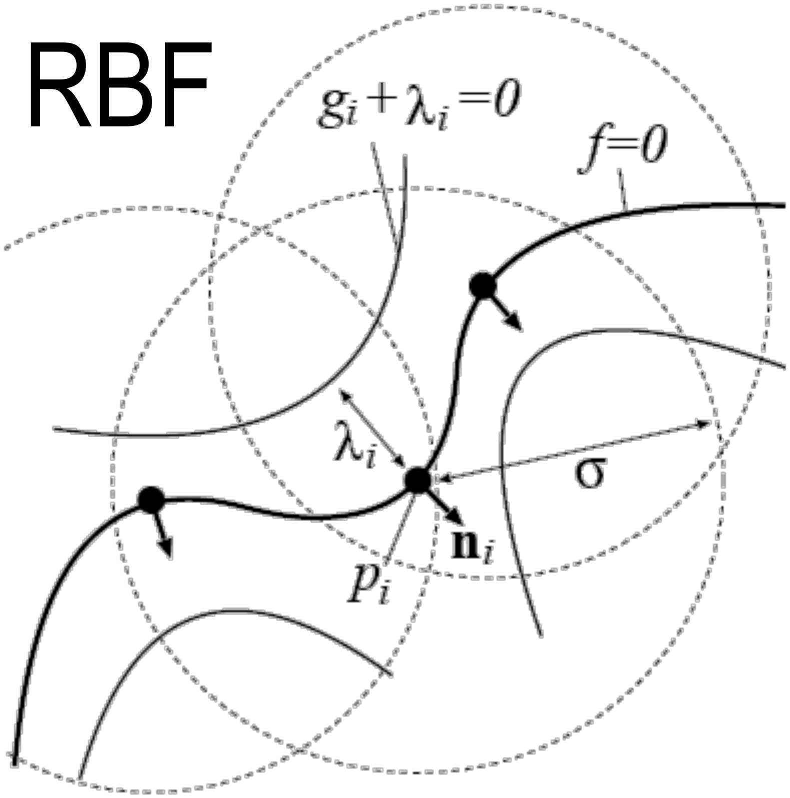
The shifting coefficients λ_i^k are found by solving the

$$f^{k-1}(\mathbf{p}_i^k) + o^k(\mathbf{p}_i^k) = 0$$

Solving Linear System

Compactly Supported RBF

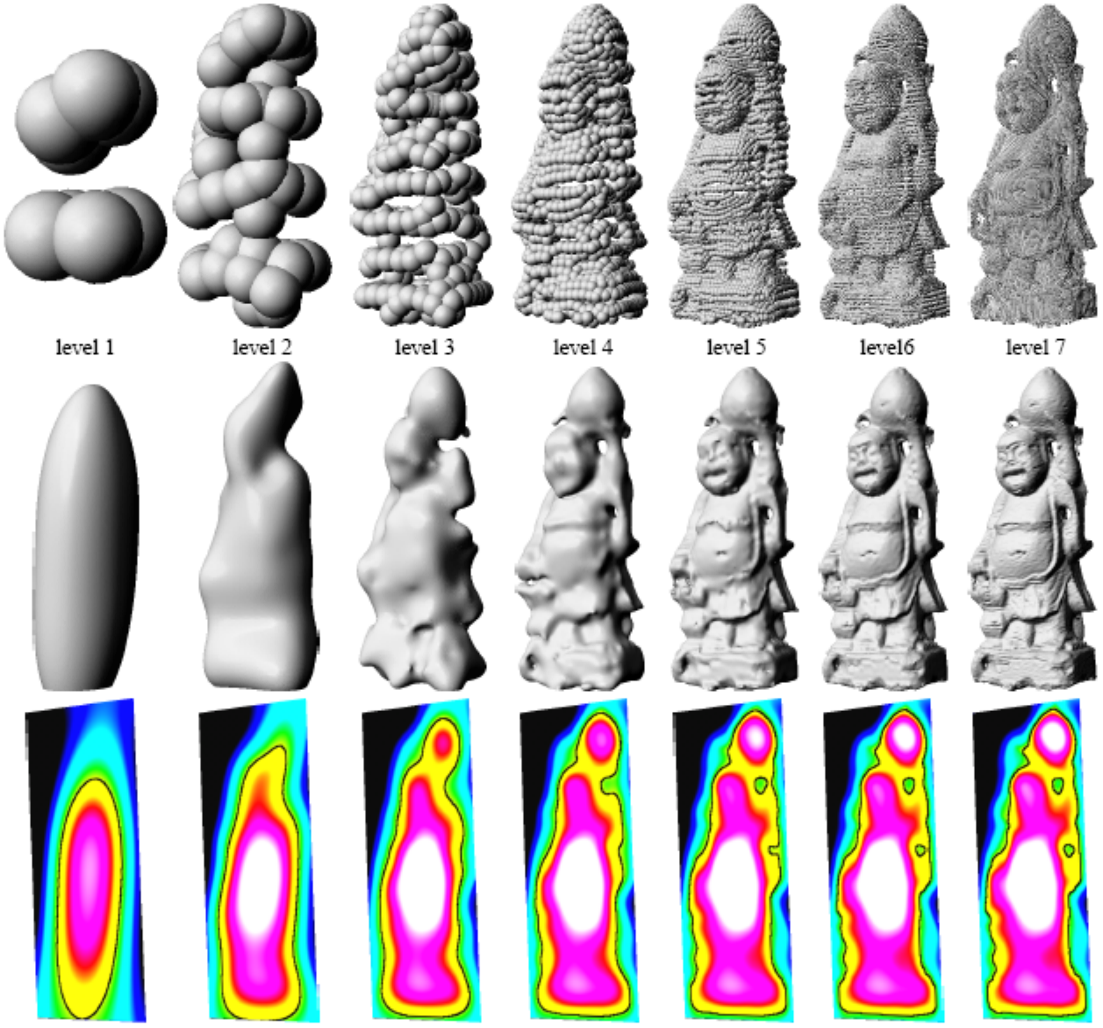




$$f(\mathbf{x}) = \sum_{\mathbf{p}_i \in \mathscr{P}} \psi_i(\mathbf{x}) = \sum_{\mathbf{p}_i \in \mathscr{P}} [g_i(\mathbf{x}) + \lambda_i]\, \phi_\sigma(\|\mathbf{x} - \mathbf{p}_i\|),$$

where $\phi_\sigma(r) = \phi(r/\sigma)$, $\phi(r) = (1-r)_+^4(4r+1)$ is Wendaland's compactly supported RBF [38], σ is its support size, and $g_i(\mathbf{x})$ and λ_i are unknown functions and coefficients

CSRBF Reconstruction



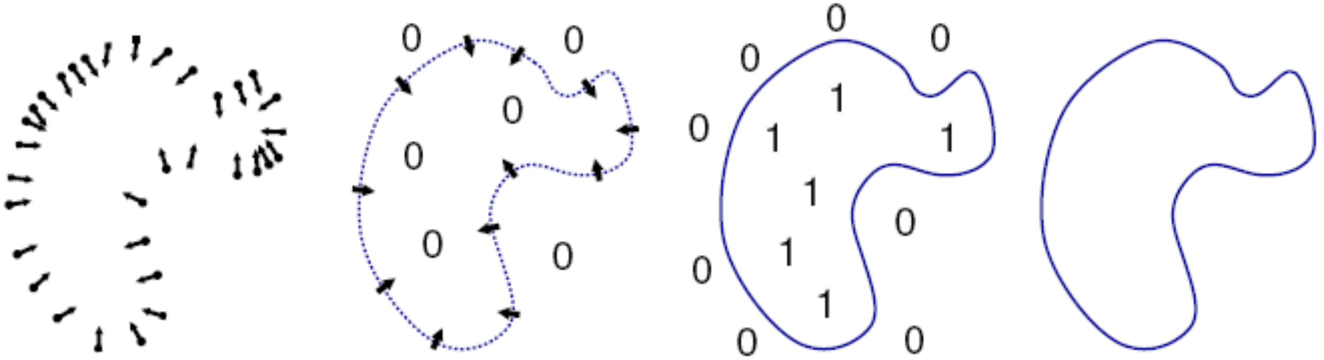
Left: a surface and feature points (ridge and ravine points) detected on it. Middle: only the feature points are kept. Right: surface reconstruction from the feature points only.

Interpolation of irregularly sampled data (73K points, 38 sec.).

Poisson Based Surface Reconstruction

- Compute a 3D indicator function: 1 for *inside* & 0 as *outside*
  - **Key insight:** there is an integral relationship between oriented points sampled from a model and its indicator function
  - Specifically, the gradient of the indicator function is a vector field that is zero almost everywhere (since the indicator function is constant), except the points near the surface, where it is equal to the inward surface normal

* The oriented point samples can be viewed as samples of the gradient of the model's indicator function



Oriented points $\vec{V}$ Indicator gradient $\nabla\chi_M$ Indicator function χ_M Surface ∂M

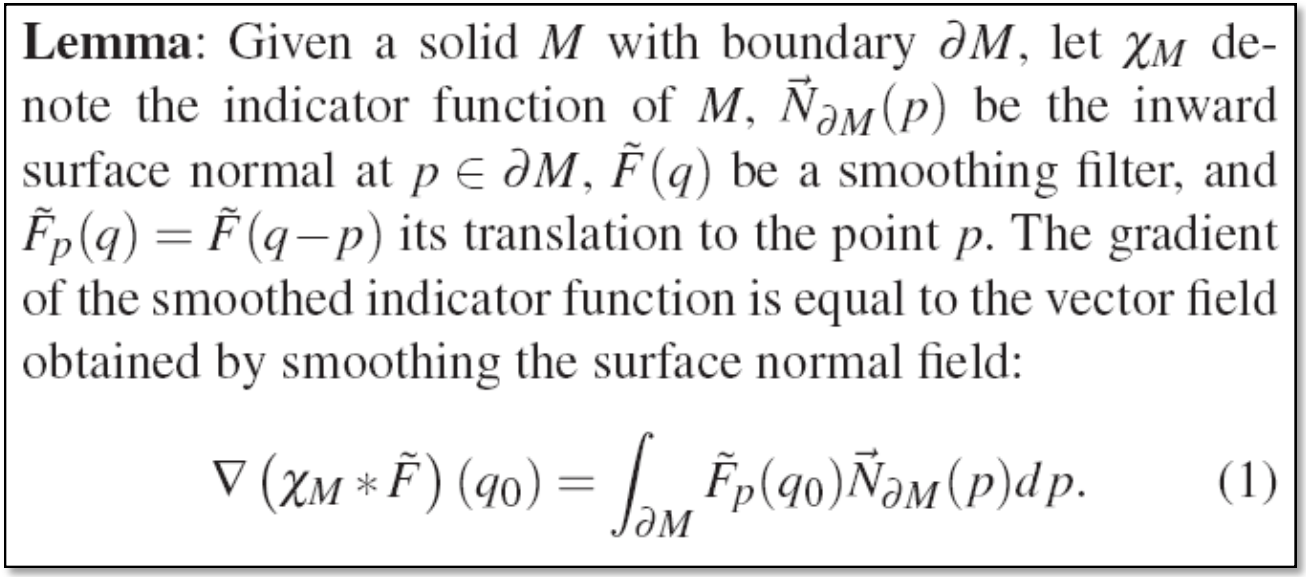
Poisson Reconstruction (cont.)

- The problem is converted to find the scalar function whose gradient best approximates a vector field defined by the samples – i.e. $\min_{\chi} ||\nabla\chi - \vec{V}||$
- If we apply the divergence operator, this variational problem transforms into a standard Poisson problem:
  - Compute the scalar function whose Laplacian (divergence of gradient) **equals** the divergence of the vector field

$$\Delta\chi \equiv \nabla \cdot \nabla\chi = \nabla \cdot \vec{V}$$

- It is a global solution but can still admit a hierarchy of locally supported functions, therefore its solution reduced to a well-conditioned sparse linear system

Numerical



Lemma: Given a solid M with boundary ∂M, let χ_M denote the indicator function of M, $\vec{N}_{\partial M}(p)$ be the inward surface normal at $p \in \partial M$, $\tilde{F}(q)$ be a smoothing filter, and $\tilde{F}_p(q) = \tilde{F}(q-p)$ its translation to the point p. The gradient of the smoothed indicator function is equal to the vector field obtained by smoothing the surface normal field:

$$\nabla\left(\chi_M * \tilde{F}\right)(q_0) = \int_{\partial M} \tilde{F}_p(q_0)\vec{N}_{\partial M}(p)\,dp. \qquad (1)$$

- Numerical Computation
- Discrete:
  - Surface is not known, how can we evaluate the integral?
  - Could we get an approximation from the input sample points?
  - Using point set *S* to partition surface into distinct patches $\mathscr{P}_s \subset \partial M$
  - We can approximate the integral over a patch by the value at point sample *s.p*, scaled by the area of the patch

$$\nabla(\chi_M * \tilde{F})(q) = \sum_{s \in S} \int_{\mathscr{P}_s} \tilde{F}_p(q)\vec{N}_{\partial M}(p)\,dp$$

$$\approx \sum_{s \in S} \underline{|\mathscr{P}_s|\, \tilde{F}_{s.p}(q)\, s.\vec{N} \;\equiv\; \vec{V}(q)}$$

Numerical Scheme for Computation

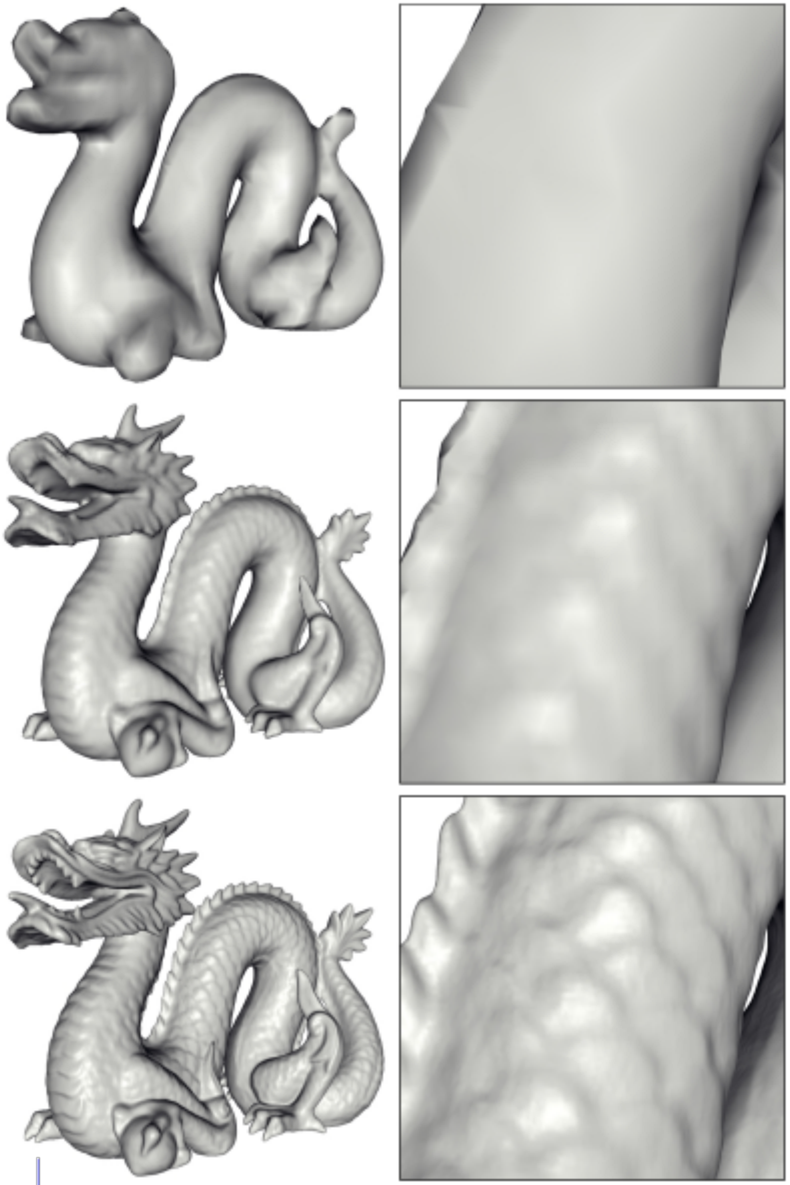
- Requirements on the filter:
  - should be sufficiently narrow so that do not over-smooth the data
  - should be wide enough so that the integral over a patch is well approximated by the value at *s.p* scaled by the patch area
- ***Candidate***: a Gaussian with variance being on the order of the sampling resolution
- Adaptive computation structure (in Octree)
  - Using the position of sample points to define the octree
  - Associate a function F_o to each node of the tree

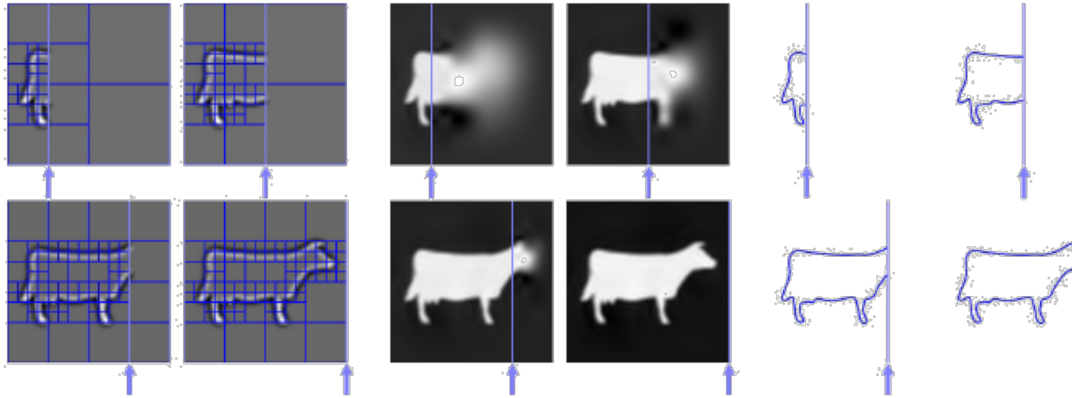
$$F_o(q) \equiv F\left(\frac{q - o.c}{o.w}\right)\frac{1}{o.w^3}.$$

where $o.c$ and $o.w$ are the center and width of node o.

Reconstruction Result

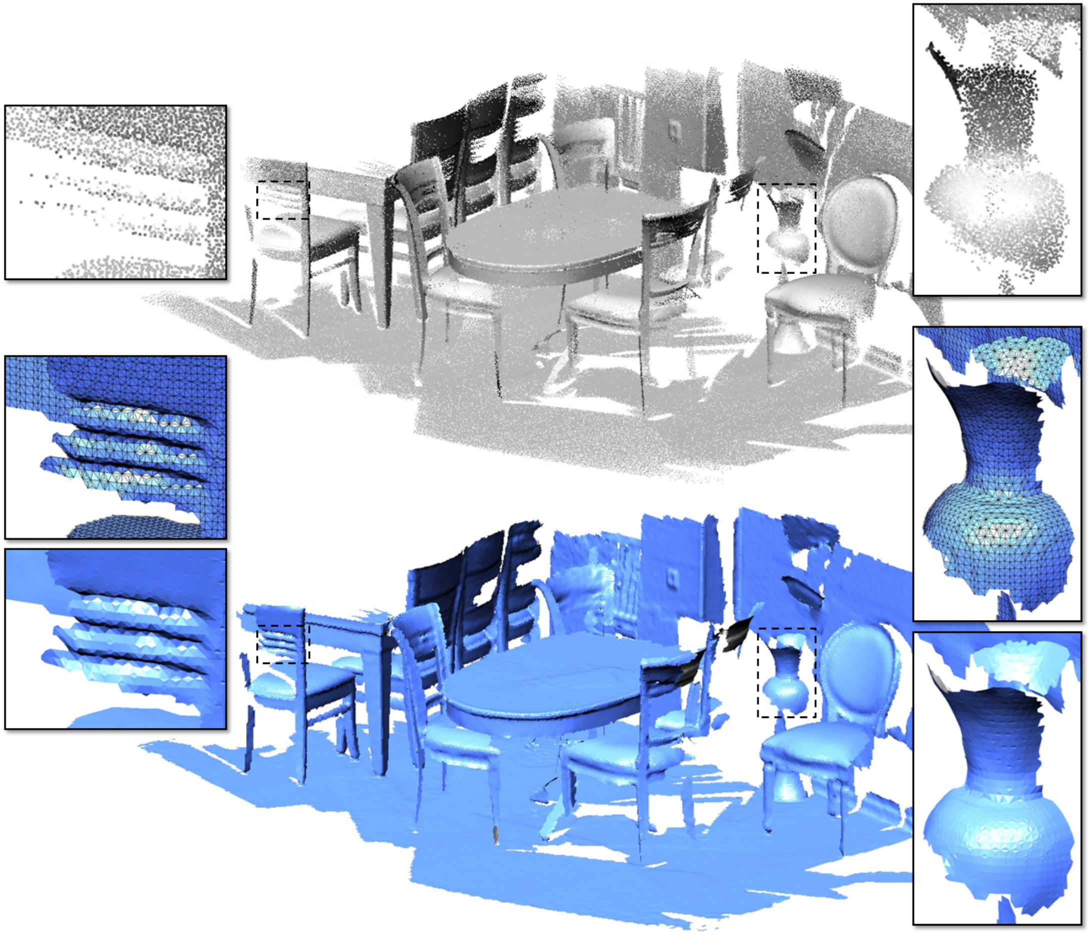
- Not sensitive to noises
- Can fill holes effectively
- Preserve normals on samples
- Can be extended to run
  - Out-of-core
  - In parallel and on GPU





[Link]

Closed-Form Formulation of HRBF-Based Surface Reconstruction



Input scenario with 922k points

Reconstruction on CPU within 5.5 sec. resulting in 313k triangles

- 17.9x faster than the state-of-the-art *Float Scaling Surface Reconstruction* (FSSR)

[Link]

Shengjun Liu, Charlie C.L. Wang, Guido Brunnett, Jun Wang. "A Closed-Form Formulation of HRBF-Based Surface Reconstruction by Approximate Solution". *Computer-Aided Design*, Special issue for SPM2016.