PCA on Human Body Database

Tsz-Ho Kwok

January 8, 2013

1 PCA on Human Body Database

Principle Component Analysis (PCA) has been used to establish statistical models for analyzing different data. In this technical report, I mainly focus on the human body database. The main advantage is that the relationship between exemplars with low variance can be discarded after PCA analysis. The full dataset does not need to be retained to represent the original examples. As a result, both the computational complexity and the data size can be greatly reduced.
Assume there are m scanned models served as exemplars, they can be listed in a matrix

$$\mathbf{H} = [\mathbf{h}_1 \ \mathbf{h}_2 \ldots \mathbf{h}_m]_{3n\times m}, \tag{1}$$

where $\mathbf{h}_i$ is a $3n \times 1$ vector which the entities are the 3D coordinates of n vertices from the mesh surface of the ith model in the database. By subtracting the average

$$\bar{\mathbf{h}} = \frac{1}{m}\sum_{i=1}^{m}\mathbf{h}_i, \tag{2}$$

The matrix $\mathbf{H}$ becomes

$$\mathbf{H} = [(\mathbf{h}_1 - \bar{\mathbf{h}}) \ (\mathbf{h}_2 - \bar{\mathbf{h}}) \ldots (\mathbf{h}_m - \bar{\mathbf{h}})]_{3n\times m} \tag{3}$$

To solve PCA on the matrix $\mathbf{H}$, I can first calculate the covariance matrix $\mathbf{C}_H = \mathbf{H}\mathbf{H}^T$, and apply eigenvalue decomposition on $\mathbf{C}_H$

$$\mathbf{C}_H\mathbf{y} = \lambda\mathbf{y} \tag{4}$$

to get the principle components as well as the variances.
However, the matrix $\mathbf{C}_H$ is $3n \times 3n$, and determining the $3n$ eigenvectors and eigenvalues is an intractable task (a human model is normally represented by few thousands or millions vertices). Fortunately, if the number of models is less than the number of vertices ($m \ll 3n$), there will be only $m-1$, rather than $3n$, meaningful eigenvectors (the remaining eigenvectors will have associated eigenvalues of zero). I can just solve for the eigenvectors of an $m \times m$ matrix instead of the $3n \times 3n$ matrix. Turk and Pentland [1] showed that if we consider the eigenvectors $\mathbf{x}$ of $\mathbf{H}^T\mathbf{H}$ such that

$$\mathbf{H}^T\mathbf{H}\mathbf{x} = \lambda\mathbf{x}, \tag{5}$$

we can premultiplying both sides by $\mathbf{H}$, and we have

$$\mathbf{H}\mathbf{H}^T\mathbf{H}\mathbf{x} = \lambda\mathbf{H}\mathbf{x}. \tag{6}$$

From which we can see that $\mathbf{H}\mathbf{x}$ are the eigenvectors of $\mathbf{C}_H = \mathbf{H}\mathbf{H}^T$.
Therefore, I can alternatively compute the transpose of the covariance, $\mathbf{C}_H^T = \mathbf{H}^T\mathbf{H}$, and apply eigenvalue decomposition on $\mathbf{C}_H^T$ as

$$\mathbf{C}_H^T\mathbf{x} = \lambda\mathbf{x}, \tag{7}$$

I can obtain m eigenvectors, and $\mathbf{x}$, which is the collection of a set of $m \times 1$ vectors. The eigenvectors $\mathbf{y}$ of the covariance matrix $\mathbf{C}_H$ can be calculated by

$$\mathbf{y} = \mathbf{H}\mathbf{x}. \tag{8}$$

where $\mathbf{y}_j$ is a $3n \times 1$ vector. The principal components of $\mathbf{H}$ is the normalized eigenvectors $\mathbf{y}_j = \mathbf{y}_j / \|\mathbf{y}_j\|$ $(j = 1, \ldots, m)$.
Another alternative that is more robust and efficient is to apply "economy size" SVD on $\mathbf{H}^T$

$$(\mathbf{H}^T)_{m \times 3n} \approx \mathbf{U}_{m \times m} \Lambda_{m \times m} (\mathbf{y}^T)_{m \times 3n} \tag{9}$$

The right singular vectors of the decomposition are the principle components, and square of singular values are the variance.
For each principal component $\mathbf{y}_j$, it is associated with an eigenvalue λ_j, which is the variance for each principle component. The principle components are sorted so that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m. \tag{10}$$

The largest variance means the corresponding component $\mathbf{y}_j$ is the most important axis to span the model space. In order to reduce the dimension of the model space, I keep the first k principal components according to the percentage of the total variance explained r by each principal component.

$$r = \frac{\lambda_1 + \lambda_2 + \ldots + \lambda_k}{\lambda_1 + \lambda_2 + \ldots + \lambda_m} \tag{11}$$

The human models in database are then projected onto k-dimensional points by

$$\mathbf{b}_i = \begin{bmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_k^T \end{bmatrix} (\mathbf{h}_i - \bar{\mathbf{h}}). \tag{12}$$

Thus, $\mathbf{H}_{3n \times m}$ is mapped into a reduced matrix $\mathbf{B}_{k \times m} = [\mathbf{b}_i]$ $(k \ll 3n)$ spanning the linear space of exemplar human bodies, named as the *reduced exemplar matrix*. Each human body is represented by k parameters, instead of $3n$ vertex positions.

References

[1] Matthew Turk and Alex Pentland, "Eigenfaces for recognition," *J. Cognitive Neuroscience*, vol. 3, no. 1, pp. 71–86, Jan. 1991.